Finding All Pure-Strategy Equilibria in Dynamic and Static Games with Continuous Strategies

Kenneth L. Judd[1] Philipp Renner[2] Karl Schmedders[3]

[1]Hoover Institution, Stanford University

[2]Dept. of Business Administration, University of Zurich

[3]Dept. of Business Administration, Univ. of Zurich and Swiss Finance Institute

ZICE Workshop – University of Zurich

September 6, 2011

swiss:finance:institute

Discrete-Time Finite-State Stochastic Games

Central tool in analysis of strategic interactions among forward-looking players in dynamic environments

Example: The Ericson & Pakes (1995) model of dynamic competition in an oligopolistic industry

Little analytical tractability

Most popular tool in the analysis: The Pakes & McGuire (1994) algorithm to solve numerically for an MPE (and variants thereof)

Applications

Advertising (Doraszelski & Markovich 2007)

Capacity accumulation (Besanko & Doraszelski 2004, Chen 2005, Ryan 2005, Beresteanu & Ellickson 2005)

Collusion (Fershtman & Pakes 2000, 2005, de Roos 2004)

Consumer learning (Ching 2002)

Firm size distribution (Laincz & Rodrigues 2004)

Learning by doing (Benkard 2004, Besanko, Doraszelski, Kryukov & Satterthwaite 2010)

Applications cont'd

Mergers (Berry & Pakes 1993, Gowrisankaran 1999)

Network externalities (Jenkins, Liu, Matzkin & McFadden 2004, Markovich 2004, Markovich & Moenius 2007)

Productivity growth (Laincz 2005)

R&D (Gowrisankaran & Town 1997, Auerswald 2001, Song 2002, Judd et al. 2011)

Technology adoption (Schivardi & Schneider 2005)

International trade (Erdem & Tybout 2003)

Finance (Goettler, Parlour & Rajan 2004, Kadyrzhanova 2005).

Need for better Computational Techniques

Doraszelski and Pakes (Handbook of IO, 2007)

"Moreover the burden of currently available techniques for computing the equilibria to the models we do know how to analyze is still large enough to be a limiting factor in the analysis of many empirical and theoretical issues of interest."

Need for better Computational Techniques II

Weintraub, Benkard, van Roy (Econometrica, 2008)

"There remain, however, some substantial hurdles in the application of EP-type models. Because EP-type models are analytically intractable, analyzing market outcomes is typically done by solving for Markov perfect equilibria (MPE) numerically on a computer, using dynamic programming algorithms (e.g., Pakes and McGuire (1994)). This is a computational problem of the highest order. [...] in practice computational concerns have typically limited the analysis [...] Such limitations have made it difficult to construct realistic empirical models, and application of the EP framework to empirical problems is still quite difficult [...] Furthermore, even where applications have been deemed feasible, model details are often dictated as much by computational concerns as economic ones."

Multiplicity of Equilibria

Besanko, Doraszelski, Kryukov, Satterthwaite (Econometrica, 2010)

"... we show that multiple equilibria in our model arise from firms' expectations regarding the value of continued play. Being able to pinpoint the driving force behind multiple equilibria is a first step toward tackling the multiplicity problem that plagues the estimation of dynamic stochastic games and inhibits the use of counterfactuals in policy analysis."

Multiplicity of Equilibria II

Besanko, Doraszelski, Kryukov, Satterthwaite (Econometrica, 2010)

"... we point out a weakness of the P-M algorithm, the major tool for computing equilibria in the literature following Ericson and Pakes (1995). Specifically, we prove that our dynamic stochastic game has equilibria that the P-M algorithm cannot compute. Roughly speaking, in the presence of multiple equilibria, "in between" two equilibria that it can compute there is one equilibrium it cannot. This severely limits its ability to provide a complete picture of the set of solutions to the model."

Outline

Polynomials

Polynomial f over the variables $z_1, \ldots, z_n$

$$f(z_1, \ldots, z_n) = \sum_{j=0}^{d} \left(\sum_{d_1+\ldots+d_n=j} a_{(d_1,\ldots,d_n)} \prod_{k=1}^{n} z_k^{d_k} \right)$$

with $a_{(d_1,\ldots,d_n)} \in \mathbb{C},\ \ d \in \mathbb{N}$

Degree of f

$$\deg f = \max_{a_{(d_1,\ldots,d_n)} \neq 0} \sum_{k=1}^{n} d_k$$

Homotopy

Continuous functions $f : \mathbb{C}^n \to \mathbb{C}^n$, $g : \mathbb{C}^n \to \mathbb{C}^n$

Homotopy from g to f is a continuous function

$$\begin{aligned} H : [0,1] \times \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (t, z) &\longmapsto H(t, z) \end{aligned}$$

such that $H(0, z) = g(z)$ and $H(1, z) = f(z)$

Polynomials in One Variable

Univariate polynomial $f(z) = \sum_{i \leq d} a_i z^i$ with $a_d \neq 0$
and so $\deg f = d$

Fundamental Theorem of Algebra: f has d complex roots
(counting multiplicities)

Simple polynomial of degree d with d distinctive complex roots

$$g(z) = z^d - 1$$

g has roots $e^{\frac{2\pi ik}{d}}$ for $k = 0, \ldots, d-1$

Homotopy $H = (1-t)g + tf$

Numerical Example

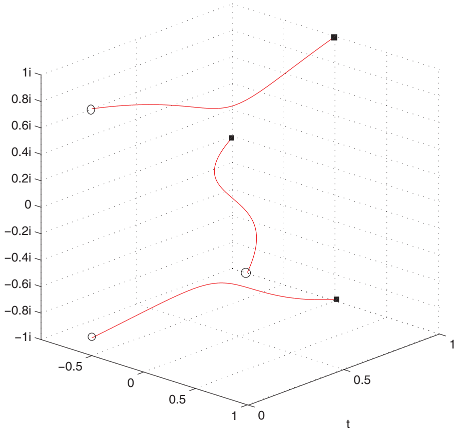
Polynomial $f(z) = z^3 + z^2 + z + 1$ with roots $-1, \ -i, \ i$

Start polynomial $g(z) = z^3 - 1$ with roots $-\frac{1}{2} - \frac{\sqrt{3}}{2}i, \ -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \ 1$

Homotopy between f and g

$$H(t, z) = (1 - t)(z^3 - 1) + t(z^3 + z^2 + z + 1)$$

Homotopy Paths



Things can go wrong

Polynomial $f(z) = 5 - z^2$ with roots $\pm\sqrt{5}$

Start polynomial $g(z) = z^2 - 1$ with roots ± 1

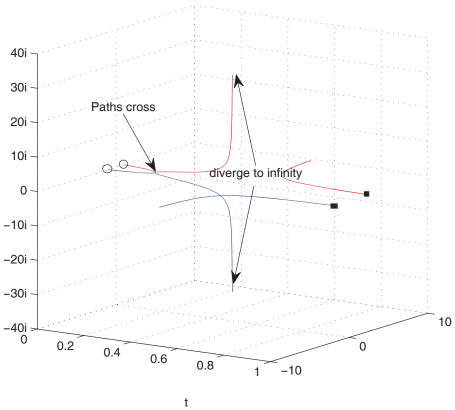
Homotopy

$$H(t, z) = t(5 - z^2) + (1 - t)(z^2 - 1) = (1 - 2t)z^2 + 6t - 1$$

$H(\frac{1}{6}, z) = \frac{2}{3}z^2$ has the double root $z = 0$, and $\det D_z H(\frac{1}{6}, 0) = 0$

$H(\frac{1}{2}, z) = 2$

Failure of Convergence



Circumventing "Bad" Points

Points of trouble

(1) Non-regular points $\det D_z H(t, z) = 0$

(2) Leading coefficient drops to zero

These points are the solution set to a system of equations

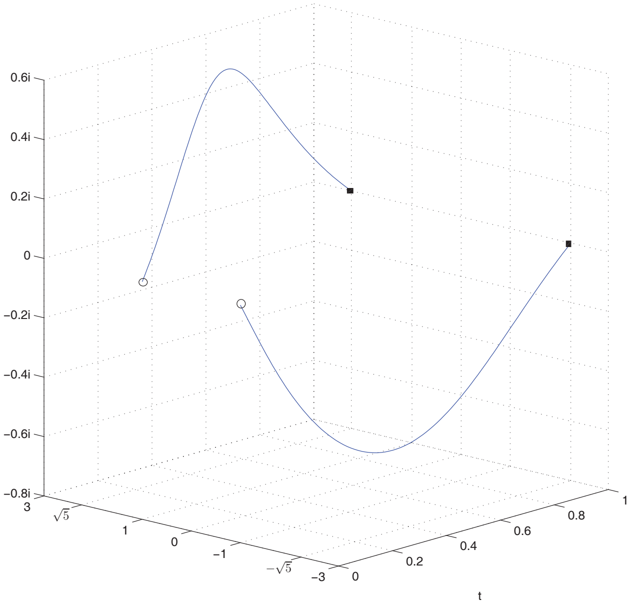
Theorem. Let $F = (f_1, \ldots, f_k) = 0$ be a system of polynomial equations in n variables, with $f_i \neq 0$ for some i. Then $\mathbb{C}^n \setminus \{F = 0\}$ is a *pathwise connected* and *dense* subset of $\mathbb{C}^n$.

Theorem implies that we can find a path between any two points without running into bad points

Gamma trick

$$H(t, z) = t(5 - z^2) + (1 - t)e^{i\gamma}(z^2 - 1)$$

Gamm



Bezout Number

Polynomial function $F = (f_1, \dots f_n) : \mathbb{C}^n \to \mathbb{C}^n$

Total degree or Bezout number of F,

$$d = \prod_i \deg f_i$$

Bezout's Theorem: system $F = 0$ has at most d isolated solutions (counting multiplicities)

Garcia and Li (1980): generic polynomial systems have exactly d distinct isolated solutions

Homotopy for Multivariate Functions

$F(z) = (f_1(z), \ldots, f_n(z)) = 0$ with $d_i = \deg f_i$

Start system $G(z) = (g_1(z), \ldots, g_n(z)) = 0$ such that

$$g_i(z) = z_i^{d_i} - 1$$

$g_i(z)$ depends only on z_i, and $\deg g_i = \deg f_i$

F and G have the same Bezout number

Homotopy with gamma trick

$$H(t, z) = e^{\gamma i}(1 - t)G(z) + tF(z)$$

For almost all $\gamma \in [0, 2\pi)$

$$|\{z | H(t_1, z) = 0\}| = |\{z | H(t_2, z) = 0\}| \quad \text{for all } t_1, t_2 \in [0, 1)$$

Convergence Theorem

For almost all $\gamma \in [0, 2\pi)$, the following properties hold.

1. The preimage $H^{-1}(0)$ consists of d regular paths.
2. Each path either diverges to infinity or converges to a solution of $F(z) = 0$ as t approaches 1.
3. If $\hat{z}$ is an isolated solution with multiplicity m, then there are m paths converging to it.
4. Paths are monotonically increasing in t.

Example

$$\begin{array}{rclcl} f_1(z_1, z_2) & = & z_1 z_2 - z_1 - z_2 + 1 & = & 0 \qquad d_1 = 2 \\ f_2(z_1, z_2) & = & (z_1)^2 z_2 - z_1 (z_2)^2 + 1 & = & 0 \qquad d_2 = 3 \end{array}$$

Start system

$$\begin{array}{rclcl} g_1(z_1, z_2) & = & (z_1)^2 - 1 & = & 0 \qquad d_1 = 2 \\ g_2(z_1, z_2) & = & (z_2)^3 - 1 & = & 0 \qquad d_2 = 3 \end{array}$$

has exactly 6 solutions

Two real and two complex solutions

$$\left(1, \frac{1}{2}(1 \pm \sqrt{5})\right) \quad \text{and} \quad \left(\frac{1}{2}(1 \pm i\sqrt{3}), 1\right)$$

Two paths diverge to infinity

Two Difficulties

Homotopy approach is intuitive, but has significant drawbacks

1. Number of finite solutions is usually much smaller than Bezout number d
   - Bezout number grows exponentially in the number of nonlinear equations
   - Most paths diverge
2. Paths diverging to infinity are a nuisance
   - Of no economic interest
   - Large computational effort
   - Require decision to truncate
   - Risk of truncating very long but converging path

Dealing with the Difficulties

Diverging paths: homogenization

compactification allows simple representation of "points at infinity"

Reduction in the number of paths

m-homogeneous Bezout number

Parameter continuation

Parameter Continuation Homotopy

Let $F(z, q) = (f_1(z, q), \ldots, f_n(z, q))$ be a system of polynomials in the variables $z \in \mathbb{C}^n$ with parameters $q \in \mathbb{C}^m$,

$$F(z, q) : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n.$$

Additionally let $q_0 \in \mathbb{C}^m$ be a point in the parameter space, where all isolated solutions z_i, $i = 1, \ldots k$ are regular. For any other set of parameters q_1 and a parameter $\gamma \in [0, 2\pi)$ define

$$\varphi(s) = sq_1 + (1 - s)q_0 + e^{i\gamma} s(1 - s)$$

Then the following statements hold for almost all $\gamma \in [0, 2\pi)$.

1. $|\{F(z, \varphi(s)) = 0\}| = k$ for all $s \in [0, 1)$.
2. The homotopy $F(z, \varphi(s)) = 0$ has k nonsingular solution paths.
3. All solution paths converge to all isolated nonsingular solutions of $F(z, \varphi(1)) = 0$.

Bertrand Price Competition

Two firms x and y produce goods x and y, prices $p_x, \; p_y$

Three types of customers with demand functions:

$$Dx1 = A - p_x \quad Dy1 = 0 \quad Dx3 = 0 \quad Dy3 = A - p_y$$

$$Dx2 = np_x^{-\sigma}\left(p_x^{1-\sigma} + p_y^{1-\sigma}\right)^{\frac{\gamma-\sigma}{-1+\sigma}} \quad Dy2 = np_y^{-\sigma}\left(p_x^{1-\sigma} + p_y^{1-\sigma}\right)^{\frac{\gamma-\sigma}{-1+\sigma}}$$

Total Demand $Dx = Dx1 + Dx2 + Dx3$

Unit cost m, thus profit $R_x = (p_x - m)$

Necessary optimality condition $MR_x = MR_y = 0$

First-order Conditions

$\sigma = 3;\ \gamma = 2;\ n = 2700;\ m = 1;\ A = 50$

First-order conditions for the two firms

$$MR_x = 50 - p_x + (p_x - 1)\left(-1 + \frac{2700}{p_x^6\left(p_x^{-2} + p_y^{-2}\right)^{3/2}} - \frac{8100}{p_x^4\sqrt{p_x^{-2} + p_y^{-2}}}\right) + \frac{2700}{p_x^3\sqrt{p_x^{-2} + p_y^{-2}}}$$

Polynomial equations ?

Polynomial System

Auxiliary variable $Z = \sqrt{p_x^{-2} + p_y^{-2}}$ yields a polynomial equation

$$0 = -p_x^2 - p_y^2 + Z^2 p_x^2 p_y^2$$

Substitute Z into denominator of MR_x and MR_y

$$0 = -2700 + 2700p_x + 8100Z^2p_x^2 - 5400Z^2p_x^3 + 51Z^3p_x^6 - 2Z^3p_x^7$$

$$0 = -2700 + 2700p_y + 8100Z^2p_y^2 - 5400Z^2p_y^3 + 51Z^3p_y^6 - 2Z^3p_y^7$$

Bezout number $d = 6 \cdot 10 \cdot 10 = 600$

Solving the System with Bertini

600 paths to track

18 real, 44 complex, 538 (truncated) infinite solutions

9 real solutions with negative values: economically meaningless

p_x	p_y
1.757	1.757
8.076	8.076
22.987	22.987
2.036	5.631
5.631	2.036
2.168	25.157
25.157	2.168
7.698	24.259
24.259	7.698

Two equilibria

Second-order conditions eliminate 5 of the 9 solutions

Check for global vs. local optimality eliminates 2 more solutions

Two equilibria

p_x	p_y
2.168	25.157
25.157	2.168

m-homogeneity: 182 paths

18 real, 44 complex, 120 (truncated) infinite solutions

Parameter Continuation Homotopy

We solved the Bertrand price game for $n = 2700$

Now we want so solve it for $n = 1000$

Parameter continuation homotopy

$$n = 2700(1-s)+(0.22334546453233+0.974739352i)s(1-s)+1000s$$

62 paths, 14 real, 48 complex solutions

Real, positive solutions

p_x	p_y
3.333	2.247
2.247	3.333
3.613	3.613
2.045	2.045
24.689	24.689

Parameter Continuation in Real Space

Parameter continuation

$n = 2700(1-s)+(0.22334546453233+0.974739352i)s(1-s)+1000s$

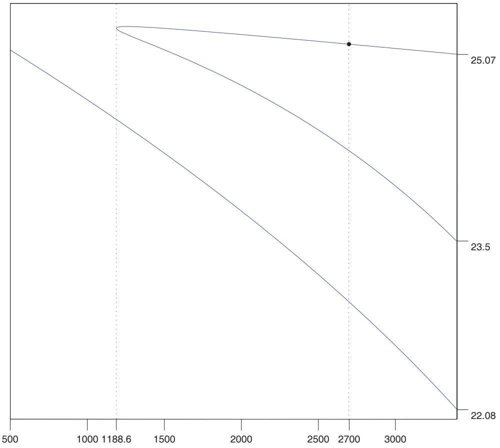
Problem: for $s \notin \{0, 1\}$ the parameter n is not a real number

Alternative approach

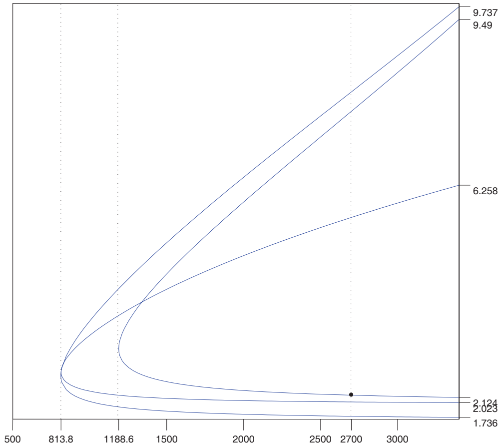
$$n = 2700(1 - s) + 1000s$$

Can we trace out the equilibrium manifold?

Parameter Continuation Homotopy



Parameter Continuation Homotopy



Static Cournot Duopoly Game

Two firms and two goods

Firm i produces good i, $i = 1, 2$

Firm i's production quantity q_i

Cost to firm i of producing q_i is $c_i(q_i; \omega_i) = \omega_i q_i$

Price of good i, $P_i(q_1, q_2) = w q_i^{-\frac{1}{\sigma}} \left(q_1^{\frac{\sigma-1}{\sigma}} + q_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\gamma-\sigma}{\gamma(\sigma-1)}}$

Firms' profit functions (revenue minus cost)

$$\begin{aligned}
\pi_1(q_1, q_2; \omega_1, \omega_2) &= q_1 \, P_1(q_1, q_2) - c_1(q_1; \omega_1) \\
\pi_2(q_1, q_2; \omega_1, \omega_2) &= q_2 \, P_2(q_1, q_2) - c_2(q_2; \omega_2)
\end{aligned}$$

Dynamic Setting

Infinite-horizon game in discrete time $t = 0, 1, 2, \ldots$

At time t firm i is in one of finitely many states,

$$\omega_{i,t} \in \Omega_i = \{1, 2, \ldots, S\}$$

State space of the game $\Omega_1 \times \Omega_2$

State of the game: production cost of two firms

Firms engage in Cournot competition in each period t

Learning-by-doing

Firms' states can change over time

Learning: current output may lead to lower production cost

Stochastic transition to state in next period

Possible transitions from state ω_i to states $\omega_i, \omega_i - 1$ in next period

Transition probability for firm i depends on q_i

$$\Pr_i[\omega_i - 1 | q_i; \omega_i] = \frac{Fq_i}{1 + Fq_i}, \quad \Pr_i[\omega_i | q_i; \omega_i] = \frac{1}{1 + Fq_i}$$

Transition Probabilities

Law of motion: State follows a controlled discrete-time, finite-state, first-order Markov process with transition probability

$$\Pr\left((\omega_1^+, \omega_2^+) | q_{1,t}, q_{2,t}; \omega_{1,t}, \omega_{2,t}\right)$$

Typical assumption of independent transitions:

$$\begin{aligned} & \Pr\left((\omega_1^+, \omega_2^+) | q_{1,t}, q_{2,t}; \omega_{1,t}, \omega_{2,t}\right) \\ = & \prod_{i=1}^{2} \Pr_i\left(\omega_i^+ | q_{i,t}; \omega_{i,t}\right) \end{aligned}$$

Objective Function

Objective of firm i is to maximize the expected NPV of future cash flows

$$\mathrm{E}\left\{\sum_{t=0}^{\infty} \beta^t \pi_i(q_{1,t}, q_{2,t}; \omega_{1,t}, \omega_{2,t})\right\}$$

with discount factor $\beta \in (0,1)$

Markov Perfect Equilibrium

Markov perfect equilibrium (MPE): pure equilibrium strategies only depend on current state and are otherwise history-independent

Firm i's strategy: $Q_i : \Omega \to \mathbb{R}_+$, $(\omega_1, \omega_2) \mapsto q_i$

$V_i(\omega)$ is the expected NPV to firm i if current state is $\omega = (\omega_1, \omega_2)$

Value function $V_i : \Omega \to \mathbb{R}$, $(\omega_1, \omega_2) \mapsto V_i(\omega)$

Firm i faces a discounted infinite-horizon dynamic programming problem, given a Markovian strategy Q_{-i} of the other firm

Bellman's optimality principle: optimal solution is again a Markovian strategy

Bellman Equation

Bellman equation for firm i is

$$V_i(\omega) = \max_{q_i} \left\{ \pi_i(q_i, Q_{-i}(\omega); \omega) + \beta \mathrm{E}\left[V_i(\omega^+) | q_i, Q_{-i}(\omega); \omega \right] \right\}$$

with Markovian strategy $Q_{-i}(\omega)$ of the other firm

Player i's strategy $Q_i(\omega)$ must satisfy

$$Q_i(\omega) = \arg\max_{q_i} \left\{ \pi_i(q_i, Q_{-i}(\omega); \omega) + \beta \mathrm{E}\left[V_i(\omega^+) | q_i, Q_{-i}(\omega); \omega \right] \right\}$$

System of equations defined above for each firm i and each state $\omega \in \Omega$ defines a pure-strategy Markov Perfect Equilibrium

Equilibrium Conditions

Unknowns $Q_i(\omega)$, $V_i(\omega)$ for each state ω

$$V_i(\omega) = \pi_i(q_i, Q_{-i}(\omega); \omega) + \beta \mathrm{E}\left[V_i(\omega^+) | q_i, Q_{-i}(\omega); \omega\right]$$

$$\frac{\partial}{\partial q_i}\left\{\pi_i(q_i, Q_{-i}(\omega); \omega) + \beta \mathrm{E}\left[V_i(\omega^+) | q_i, Q_{-i}(\omega); \omega\right]\right\} = 0$$

First-order conditions are necessary and sufficient

Assumptions ensure interior solutions $q_i > 0$

Transformation into polynomial system of equations

Two equations per firm per state, total of $4S^2$ equations

Simplification

Nature of transitions induces a partial order on the state space Ω

Instead of one system with $4S^2$ equations solve S^2 systems of 4 equations each

Solve games for

1) lowest-cost state $(1, 1)$ (static Cournot game)

2) for states $(\omega_1, 1)$ for $\omega_1 = 2, \dots, S$ and for states $(1, \omega_2)$ for $\omega_2 = 2, \dots, S$

3) for states $(\omega_1, 2)$ for $\omega_1 = 2, \dots, S$ and for states $(2, \omega_2)$ for $\omega_2 = 2, \dots, S$

and so on ...

Numerical Example in Bertini

$\sigma = 2$, $\gamma = 3/2$, $w = 100/3$, $F = 1/5$, $\beta = 0.95$.

After transformations: 6 equations in 6 unknowns

state	Bezout #	m-hom. #	time
$(1,1)$	216	44	4 sec
$(1,\omega_2)$	360	140	1 min
$(2,2)$	1176	364	5 min

Identical degree structure for all states (ω_1, ω_2) with $\omega_1, \omega_2 \geq 2$

Parameter continuation: 152 paths in 25 sec

Quantities and Value Function of Firm 1

$\omega_1 \setminus \omega_2$	5		4		3	
5	7.202	874	7.108	861	7.009	851
4	8.850	939	8.748	925	8.620	913
3	11.475	996	11.385	982	11.233	969
2	16.921	1042	16.840	1027	16.699	1014
1	38.228	1072	38.171	1057	38.056	1043

$\omega_1 \setminus \omega_2$	2		1	
5	6.889	843	6.626	838
4	8.464	905	8.137	899
3	11.016	959	10.573	953
2	16.401	1003	15.714	997
1	37.773	1032	36.600	1025

Summary and Outlook

All-solution homotopy methods for polynomial systems of equations have applications in economics

Find all solutions to equilibrium equations

Computational approach to "proving" uniqueness

Drawback: "curse of dimensionality" as number of equations increases

Parameter-continuation homotopies greatly reduce number of paths

Parallelization