Error Analysis in Dynamic Models II.

by Adrian Peralta-Alva

Federal Reserve Bank of Saint Louis

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Markov Equilibria in Macroeconomics

Based on Krueger, Dirk and Felix Kubler, 2009. "Markov Equilibria in Macroeconomics," The New Palgrave Dictionary of Economics, Steven N. Durlauf and Lawrence E. Blume. Eds. Palgrave Macmillan

Overview

- We will denote histories of shocks up to period t by $z^t = z_0, z_1, \ldots, z_t$.
- **Sequential Equilibria (SE)** are sequences of functions mapping histories of realized shocks into prices and allocations such that all agents solve their intertemporal optimization problems and all markets clear.
- Under fairly mild conditions such equilibria exist.
- For computational purposes we need alternative characterizations.

Markov Equilibria

- Is characterized by a state-space, a policy function, and a transition function.
- The policy maps the state today into current endogenous choices and prices.
- The transition maps the state today into a probability distribution over states tomorrow.
- We say Sequential Equilibria has a recursive representation if the sequences generated by such policy and transition maps conform a SE.
- SE may be Markovian, but in what state space?.
- For computation, we want the minimal dimension.

State space

- In very few models, the vector of exogenous shocks maybe enough to characterize equilibria (e.g. Lucas (1988) asset pricing model).
- More generally, endogenous variables have to be included for SCE to have a recursive representation.
- Exogenous shocks and all endogenous variables that are predetermined today are the standard, or minimal state space.
- Equilibria on the minimal state space will be called 'simple Markov equilibria'

Pareto Optimal Economies

- If SE can be recasted as a planner's problem then SE will be simple Markov iff the planner's problem solution is simple Markov.
- This is commonly done by writing the planner's problem as a functional equation and showing that the associated policy function yields a solution to the planner's sequential problem.
- If the value function iteration operator is a contraction, then a unique value function solves the social planner's functional equation.
- More general fixed point theorems establish existence of the planner's functional equation when the contraction property cannot be established.

Example

Planner's problem is

$$\sup_{\{x_t\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

s.t.

$$x_{t+1} \in \Gamma(x_t),\, x_0 \, given.$$

in the one sector growth model $F(k_t, k_{t+1}) = u(Ak_t^{\alpha} + (1-\delta)k_t - k_{t+1})$; and $\Gamma(k_t) = [0, Ak_t^{\alpha} + (1-\delta)k_t]$.

The corresponding functional equation is

$$V(x) = \sup_{y \in \Gamma(x)} F(x,y) + \beta V(y)$$

Numerical Methods

- The principle of optimality implies existence of simple Markov equilibria, and value function iteration provides a reliable approximation method (our previous lecture covered accuracy properties).
- The existence of simple Markov gives foundations to methods based on searching for a function (with domain the minimal state) that solves FOCs.
- In our previous lecture we employed dynamic programming and iterative arguments to derive error bounds for any policy function (provided the model has a simple Markov equilibrium).

Non existence of simple Markov

- In models where the first welfare theorem is not applicable, in models with infinitely many agents (OLG), or in models with strategic interaction the existence of simple Markov equilibria cannot be guaranteed.
- See Santos (2002 JET), Krebs (2004, JET), Kubler and Schmedders (2002, Macro. dyn.) and Kubler and Polemarchakis (2004, ET) for simple counter examples.

Recovering a recursive structure

- Kydland and Prescott (1980) first noted the time inconsistency issue (a Markovian policy on a minimal state space may not solve the sequential problem) in an optimal taxation (Ramsey) excercise.
- They also pioneered the idea that recursivity can be recovered on a state space constituted by the minimal one, together with last period's marginal utility.
- Duffie et al (ECTA 1994) show that the Markov property holds for a fairly general class of models on a comprehensive state space that includes **all current endogenous and exogenous** variables.
- We will call equilibria that is Markov on a state space larger than the minimal one **generalized Markov equilibria.**

Issues to be discussed

- Aren't these just technicalities. Suppose we apply a standard method and using the computer we "find" a simple Markov equilibrium. Does this common practice suggest we should not care about such technicalities?
- Simple examples show serious inaccuracies and biases may emerge once we use methods that do not rely on solid theoretical grounds.
- How can we compute these models? How to simulate and estimate?

Problems in the Simulation of Models with Frictions

Based on Peralta-Alva, Adrian and Manuel S. Santos, 2010. "Problems in the Numerical Simulation of Models with Heterogeneous Agents and Economic Distortions," Journal of the European Economic Association, Vol. 8 (2-3), Pages 617-625

Main problems

- Main issue is numerical tractability.
- The welfare theorems do not hold and the characterization of the solution is much harder.
- The principle of optimality may not hold.
- A recursive structure may be recovered but only in an enlarged state space; furthermore, this Markovian mapping may not be continuous.

Neoclassical Growth Model with Taxes

Representative household's problem (taking taxes, transfers and prices as given):

$$\max \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

$$s.t.$$

$$c_t + k_{t+1} \leq \pi_t + (1-\tau_t)\, r_t k_t + T_t$$

$$k_0 \text{ given, } 0 < \beta < 1,$$

$$c_t \geq 0,\ k_{t+1} \geq 0 \text{ for all } t \geq 0.$$

Taxes are functions of the aggregate capital stock K_t. All tax revenues are rebated back to the consumer as lump-sum transfers T_t.

Parameterization

Let

$$f(K) = K^{1/3},\ \beta = 0.95.$$

and consider the following piecewise linear tax function:

$$\tau(K) = \begin{cases} 0.10 & \text{if } \ K \leq 0.160002 \\ 0.05 - 10(K - 0.165002) & \text{if } \ 0.160002 \leq K \leq 0.170002 \\ 0 & \text{if } \ K \geq 0.170002. \end{cases}$$

Santos (2002, Prop. 3.4) shows that a continuous Markov equilibrium fails to exist. For this specification of the model, there are three steady states: The middle steady state is unstable and has two complex eigenvalues while the other two steady states are saddle-path stable.

Solving by standard methods

Find a function of the form

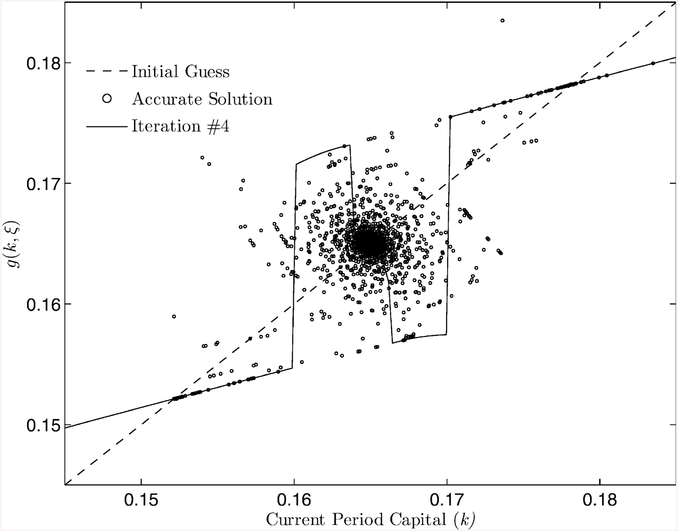
$$k_{t+1} = g(k_t, \xi),$$

where g belongs to a finite dimensional space of continuous functions as defined by a vector of coefficients ξ. We obtain an estimate for ξ by forming a discrete system of Euler equations over as many grid points k^i as the dimensionality of the ξ:

$$u'(k^i, g(k^i, \xi)) =$$

$$= \beta u'(g(k^i, \xi), g(g(k^i, \xi), \xi)) \cdot \left[f'(g(k^i, \xi))(1 - \tau(g(k^i, \xi))) \right]$$

Solution vs Approximation



Solution vs Approximation

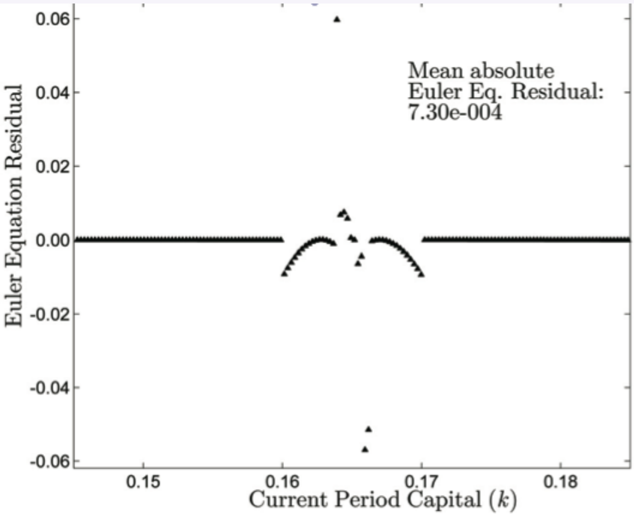


Figure 2: Euler Equation Residual of Continuous Approximation.

A Monetary OLG economy

One good, representative agent lives 2 periods. The only asset is fiat money, and initial old agent is endowed with the existing money supply M. Let P_t be the price level at time t. An agent born in period t chooses consumption c_{1t} when young, c_{2t+1} when old, and money holdings M_t to solve

$$\max u(c_{1t}) + \beta v(c_{2t+1})$$

subject to

$$\begin{aligned} c_{1t} + \frac{M_t}{P_t} &= e_1, \\ c_{2t+1} &= e_2 + \frac{M_t}{P_{t+1}}. \end{aligned}$$

Equilibrium

A sequential competitive equilibrium for this economy is a sequence of prices $(P_t)_{t \geq 0}$, and sequences of consumption and money holdings $\{c_{1t}, c_{2t+1}, M_t\}_{t \geq 0}$ such that solve the household's problem and markets clear:

$$c_{1t} + c_{2t} = e_1 + e_2, \text{ and } M_t = M \text{ for all } t.$$

Equilibrium

A sequential competitive equilibrium can be characterized by:

$$\frac{1}{P_t} u'\left(e_1 - \frac{M}{P_t}\right) = \frac{1}{P_{t+1}} \beta v'\left(e_2 + \frac{M}{P_{t+1}}\right).$$

Let $b_t = M/P_t$ be real money balances at t. Then,

$$b_t u'(e_1 - b_t) = b_{t+1} \beta v'(e_2 + b_{t+1}).$$

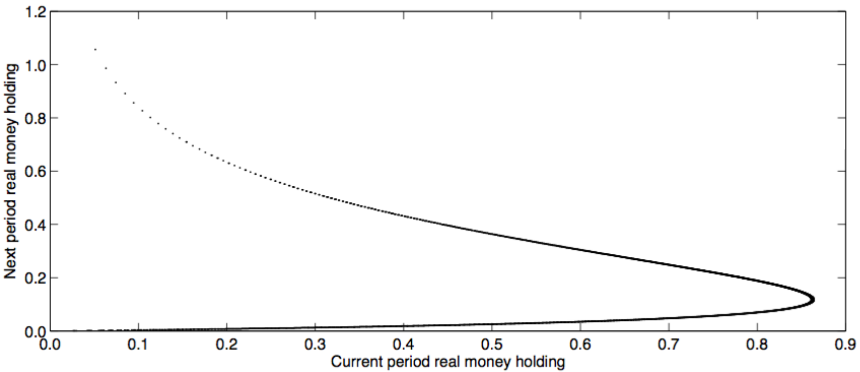
Parameterization

Let us now restrict attention to the following parameterizations:

$$u(c) = c^{0.45},\ v(c) = -\frac{1}{7}c^{-7},\ \beta = 0.8,$$

$M = 1$, $e_1 = 2$, and $e_2 = 2^{6/7} - 2^{1/7}$. For this simple example, the offer curve is backward bending.

Solution

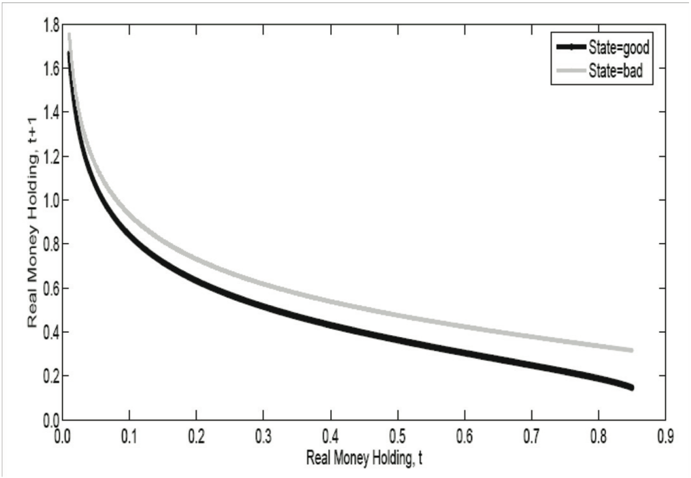


Next period real money holding
Current period real money holding
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

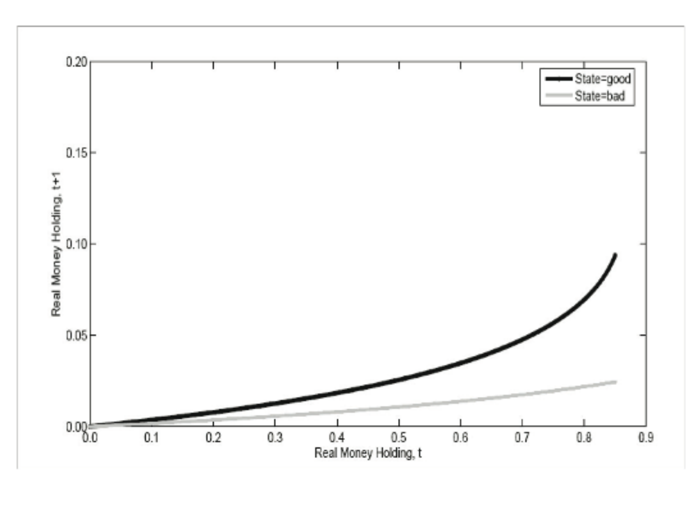
Standard Methods

- A common practice in OLG is to search for an equilibrium given by $b' = g(b, \xi)$.
- We applied such a procedure to our model. Depending on the initial guess, we find that either the upper or the lower part of the offer curve would emerge as a fixed point.
- This strong dependence on initial conditions is rather undesirable.

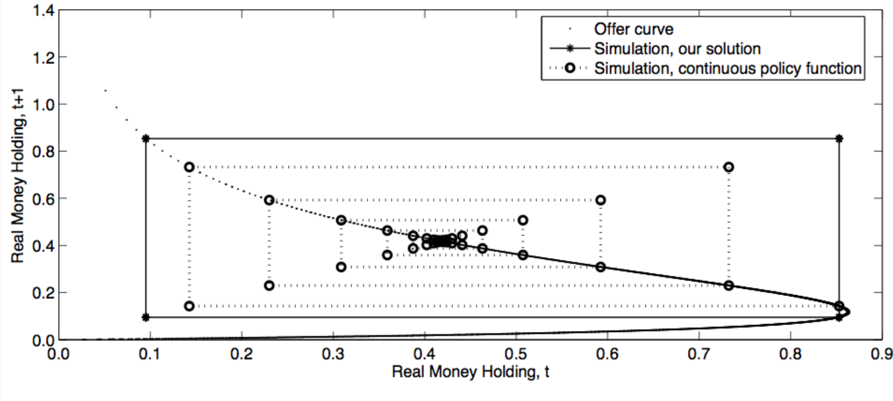
Standard Methods (Initial condition 1)



Standard Methods (Initial condition 2)



Dynamics under true and standard solution



A Stochastic OLG Economy with Bonds

- Two possible values for shocks, $S = \{s_1, s_2\}$; state histories are $\sigma_t = (s_1, .., s_t)$.
- Two agents that leave for 2 periods, two goods. Agent i born in period t is denoted by (i, σ_t).
- The only available asset is a one-period risk-free bond (in zero net supply). Its price is $q(\sigma_t)$. Agent i holdings are denoted by θ^i.
- In the first period of life of each agent endowments are stochastic (and depend only on the current state s_t), while in the second period they are deterministic.

Framework

Agent 1 solves

$$\max -\frac{1024}{\left(x_1^1\left(\sigma_t\right)\right)^4}+E_{s_{t+1}|\sigma_t}\left[-\frac{1024}{\left(x_1^1\left(\sigma_t s_{t+1}\right)\right)^4}-\frac{1}{\left(x_2^1\left(\sigma_t s_{t+1}\right)\right)}\right]$$

$$s.t.$$

$$e_1^1(s_t)=q\left(\sigma_t\right)\theta^1\left(\sigma_t\right)+x_1^1\left(\sigma_t\right)$$

$$12+p\left(\sigma_t s_{t+1}\right)+\theta^1\left(s_t\right)=x_1^1\left(\sigma_t s_{t+1}\right)+p\left(\sigma_t s_{t+1}\right)x_2^1\left(\sigma_t s_{t+1}\right).$$

While agent 2 solves

$$\max -\frac{1}{\left(x_1^2\left(\sigma_t\right)\right)^4}+E_{s_{t+1}|\sigma_t}\left[-\frac{1}{\left(x_1^2\left(\sigma_t s_{t+1}\right)\right)^4}-\frac{1024}{\left(x_2^2\left(\sigma_t s_{t+1}\right)\right)}\right]$$

$$s.t.$$

$$e_1^2(s_t)=q\left(\sigma_t\right)\theta^2\left(\sigma_t\right)+x_1^2\left(\sigma_t\right)$$

$$1+12p\left(\sigma_t s_{t+1}\right)+\theta^2\left(s_t\right)=x_1^2\left(\sigma_t s_{t+1}\right)+p\left(\sigma_t s_{t+1}\right)x_2^2\left(\sigma_t s_{t+1}\right).$$

Equilibrium

A competititve equilibrium is constituted by stochastic sequences of prices $\{\hat{q}(\sigma_t), \hat{p}(\sigma_t)\}_{\sigma_t}$ and portfolio and consumption allocations $\left\{\hat{x}^i, \hat{\theta}^i\right\}_{\sigma_t}$ such that:

1. Portfolio and consumption allocations solve the individual optimization problems taken prices as given.
2. Financial markets clear, so that $\theta^1 = \theta^2$, and goods markets clear, so that

$$\begin{aligned} 13 + e_1^{1,\sigma_{t+1}}(s_{t+1}) + e_1^{2,\sigma_{t+1}}(s_{t+1}) &= \sum_i x_1^{i,\sigma_t}(\sigma_{t+1}) + \sum_i x_1^{i,\sigma_{t+1}}(\sigma_{t+1}) \\ 13 &= \sum_i x_2^{i,\sigma_t}(\sigma_{t+1}) \end{aligned}$$

Standard Approach

Standard computational methods search for continuous, time invariant, functions $f_x, f_{x'}, f_\theta, f_p, f_q$ such that:

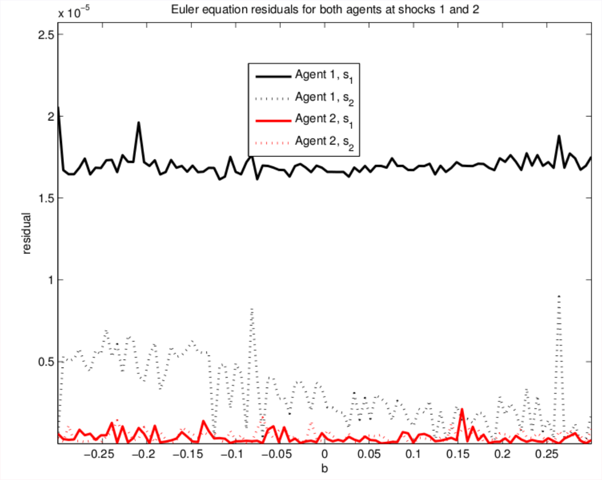
$$
\begin{aligned}
\hat{x}_1^{i,\sigma_t}(\sigma_t) &= f_x\left(s_t, \theta^{1,\sigma_{t-1}}\right) \\
\hat{x}_l^{i,\sigma_{t-1}}(\sigma_t) &= f_{x'}\left(s_t, \theta^{1,\sigma_{t-1}}\right) \\
\hat{\theta}^{1,\sigma_t} &= f_\theta\left(s_t, \theta^{1,\sigma_{t-1}}\right) \\
\hat{p}(\sigma_t) &= f_\theta\left(s_t, \theta^{1,\sigma_{t-1}}\right) \\
\hat{q}(\sigma_t) &= f_q\left(s_t, \theta^{1,\sigma_{t-1}}\right).
\end{aligned}
$$

Equilibrium

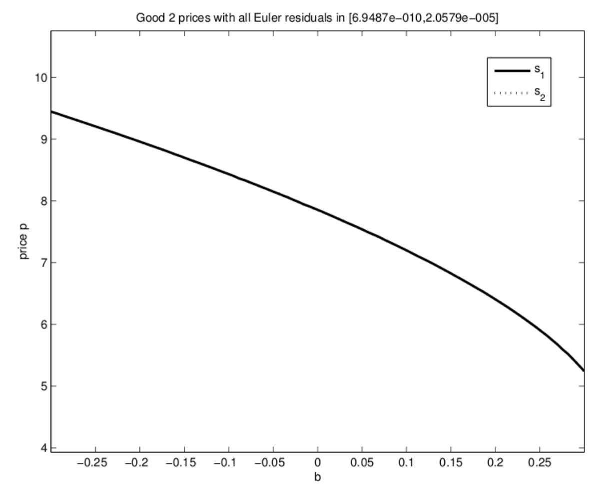
1. $\theta^1 = 0$ at all σ_t.
2. Given node σ_{t-1} with $s_{t-1} = s_1$, we have that for all successors of σ_{t-1}, namely $\sigma_t = \sigma_{t-1}s_1$ or $\sigma_t = \sigma_{t-1}s_2$:
   $\left(x_1^{1,\sigma_{t-1}}(\sigma_t), x_2^{1,\sigma_{t-1}}(\sigma_t)\right) = (10.4, 2.6)$,
   $\left(x_1^{2,\sigma_{t-1}}(\sigma_t), x_2^{2,\sigma_{t-1}}(\sigma_t)\right) = (2.6, 10.4)$, and $p(\sigma_t) = 1$.
3. Given node σ_{t-1} with $s_{t-1} = s_2$, we have that for all successors of σ_{t-1}, namely $\sigma_t = \sigma_{t-1}s_1$ or $\sigma_t = \sigma_{t-1}s_2$:
   $\left(x_1^{1,\sigma_{t-1}}(\sigma_t), x_2^{1,\sigma_{t-1}}(\sigma_t)\right) = (8.4, 1.4)$,
   $\left(x_1^{2,\sigma_{t-1}}(\sigma_t), x_2^{2,\sigma_{t-1}}(\sigma_t)\right) = (4.6, 11.6)$, and $p(\sigma_t) = 7.9$.

Hence, knowledge of the current shock, s_t, and wealth distribution are not enough to characterize consumption of the old.

Using the computer we "found" simple Markov



Using the computer we obtained very biased results



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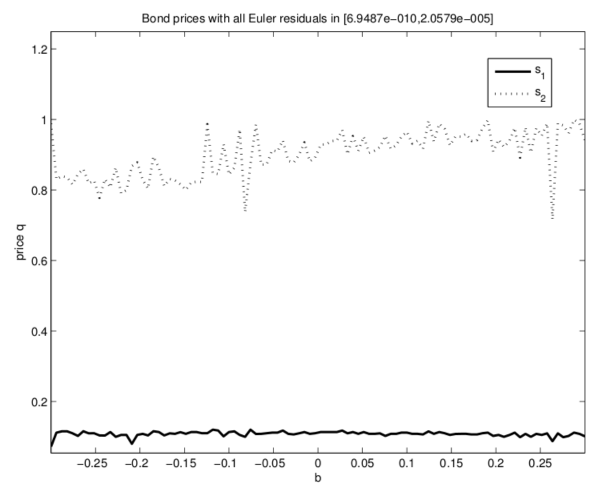

Computing Models with Frictions

Based on Feng, Zhigang, Miao, J., Peralta-Alva, A. and Santos, M. "Numerical Simulation of Nonoptimal Dynamic Equilibrium Models," Working Paper.

Motivation

- Since Kydland and Prescott (cf JET 1980) it is well known that problems that are not recursive on the natural state space may be recursive on an enlarged state space.
- Indeed, Duffie, D., J. Geanakoplos, A. Mas-Colell and A. McLennan (ECTA 1994) show that if all endogenous variables are included as states then recursive equilibria exists in a very general class of models.
- However, their results do not tell us how to find such equilibria, and apply only to models with exogenous constraints.
- We develop the first reliable method to solve models with frictions with either endogenous or exogenous constraints.

Asset pricing with incomplete markets and borrowing constraints

The representative household of type $i = 1, ..., N$, solves the following problem:

$$\max E_0 \sum_{t=0}^{\infty} \sum_{\eta^t} \beta^t \pi(\eta^t) u(c^i(\eta^t))$$

$s.t.$

$$c^i(\eta^t) + V(\eta^t)[s^i_{t+1} - s^i_t(\eta^t)] \leq$$
$$s^i(\eta^t) d(\eta^t) + e^i(\eta^t)$$
$$\kappa(\eta^t) \leq s^i_{t+1}$$

SCE characterized by

$$
\begin{aligned}
FOCs: \quad & u_{c^i(\eta^t)} V(\eta^t) = \beta \sum \pi(\eta^{t+1}|\eta^t) u_{c^i(\eta^{t+1})} [V(\eta^{t+1}) + d(\eta^{t+1})] \\
& + \tfrac{\mu(\eta^t)}{\beta^t \pi(\eta^t)} \\
BC: \quad & c^i(\eta^t) + V(\eta^t)[s^i_{t+1} - s^i_t(\eta^t)] \leq \\
& s^i(\eta^t) d(\eta^t) + e^i(\eta^t) \\
MC: \quad & \textstyle\sum_i s^i_{t+1} = 1
\end{aligned}
$$

Towards a recursive characterization

- The minimal state space consists of the distribution of shares (or wealth) , and the shocks (dividends and endowments).
- SCE may not have a recursive characterization on the minimal state.

Towards a recursive characterization

Of course, we are only interested in specifications for the model such that SCE exists. Suppose we knew at each η^t the set of possible equilibrium values

$$m(\eta^t) \equiv u_{\hat{c}(\eta^t)}[\hat{V}(\eta^t) + \hat{d}(\eta^t)],$$

For each given initial condition (η_0, s_0) we construct the equilibrium correspondence $W(\eta_0, s_0)$ as the set of equilibrium values for m from such initial condition. This together with operator B are the key ingredients of our approach.

Towards a recursive characterization

Recall

$$m(\eta^t) \equiv u_{\hat{c}(\eta^t)}[\hat{V}(\eta^t) + \hat{d}(\eta^t)],$$

Given η_0, s_0, $\tilde{m} \in W(\eta_0, s_0)$ operator B can be defined by $\tilde{m} \in BW(\eta_0, s_0)$ iff there exist $\tilde{V}, \tilde{c}, \tilde{s}_+, \tilde{\mu}$ and for each possible value of shocks next period $\tilde{m}(\eta_+) \in W(\eta_+, \tilde{s}_+)$:

$$\begin{array}{lc} 1) & u_{\tilde{c}^i}\tilde{V} = \beta \sum \pi(\eta_+|\eta_0)\tilde{m}^i(\eta_+) + \tilde{\mu}^i \\ & \text{with } \tilde{\mu}^i > 0\, only \text{if } \kappa^i(\eta) = \tilde{s}^i_+ \\ 2) & \tilde{c}^i + \tilde{V}[\tilde{s}^i_+ - \tilde{s}^i] = d\tilde{s}^i + e^i \\ 3) & \sum_i \tilde{s}^i = 1 \end{array}$$

A recursive characterization

Claim: Any fixed point of operator B, $W^* = BW^*$ can be used to characterize SCE recursively, on the state space η, s, m.
For it, take $\eta_0, s_0,\ \tilde{m} \in W(\eta_0, s_0)$ as given. Then, there is $\tilde{V}_0, \tilde{c}_0, \tilde{s}_1 = \tilde{s}_+, \mu_0 = \tilde{\mu}$,and for each possible value of the shock $\eta_1 = \eta_+$values $\tilde{m_1} \in W(\eta_1, s_1)$ that satisfy 1)-3) above. Now take each one of the possible shocks of period 1, together with its corresponding $\tilde{s}_1, \tilde{m_1} \in W(\eta_1, \tilde{s}_1)$. By definition of the equilibrium correspondence, there must exist $\tilde{V}_1, \tilde{c}_1, \tilde{s}_2, \tilde{\mu}_1$that satisfy 1)-3) above. From the defintion of m_1, and 1) at period 0 we have

$$u_{\tilde{c}_0^i} \tilde{V}_0 = \beta \sum \pi(\eta_1|\eta_0) u_{\tilde{c}_1^i(\eta_1)}[\tilde{V}(\eta_1) + d(\eta_1)],$$

proceeding in this fashion we can generate sequences of prices and allocations that satisfy the Euler equations, individual budget constraints, and market clearing. Such conditions fully characterize equilibria for this model.