Mixed Integer Nonlinear Programming (MINLP)

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Overview

Part I

Introduction, Applications, and Formulations

The Problem of the Day

Mixed Integer Nonlinear Program (MINLP)

$$\begin{cases} \underset{x,y}{\text{minimize}} & f(x, y) \\ \text{subject to} & c(x, y) \leq 0 \\ & x \in X, \; y \in Y \text{ integer} \end{cases}$$

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- X, Y polyhedral sets, e.g. $Y = \{y \in [0, 1]^p \mid Ay \leq b\}$

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- f, c smooth (convex) functions
- X, Y polyhedral sets, e.g. $Y = \{y \in [0,1]^p \mid Ay \leq b\}$
- $y \in Y$ integer $\Rightarrow$ hard problem
- f, c not convex $\Rightarrow$ very hard problem

Why the MI?

- We can use 0-1 (binary) variables for a variety of purposes
  - Modeling yes/no decisions
  - Enforcing disjunctions
  - Enforcing logical conditions
  - Modeling fixed costs
  - Modeling piecewise linear functions

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  - Enforcing logical conditions
  - Modeling fixed costs
  - Modeling piecewise linear functions
- If the variable is associated with a physical entity that is indivisible, then it must be integer
  - Number of aircraft carriers to to produce. Gomory's Initial Motivation

A Popular MINLP Method

Dantzig's Two-Phase Method for MINLP

Adapted by Leyffer and Linderoth

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1. Convince the user that he or she does not wish to solve a mixed integer nonlinear programming problem at all!

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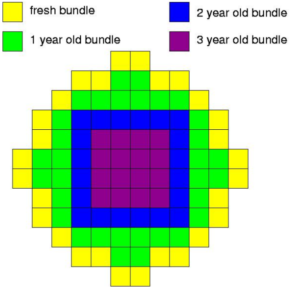
2. Otherwise, solve the continuous relaxation (NLP) and round off the minimizer to the nearest integer.

- For $0-1$ problems, or those in which the $|y|$ is "small", the continuous approximation to the discrete decision is not accurate enough for practical purposes.
- Conclusion: MINLP methods must be studied!

Example: Core Reload Operation (Quist, A.J., 2000)

- max. reactor efficiency after reload subject to diffusion PDE & safety
- diffusion PDE $\simeq$ nonlinear equation $\Rightarrow$ integer & nonlinear model
- avoid reactor becoming sub-critical



Example: Core Reload Operation (Quist, A.J., 2000)

- max. reactor efficiency after reload subject to diffusion PDE & safety
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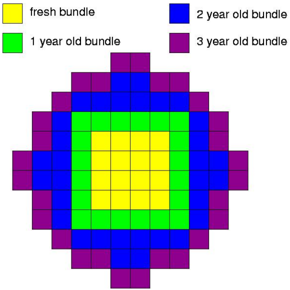
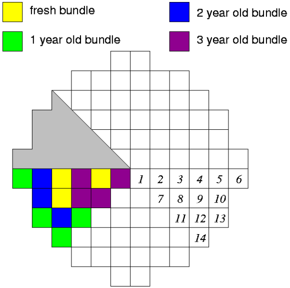


Example: Core Reload Operation (Quist, A.J., 2000)

- look for cycles for moving bundles:
  e.g. $4 \to 6 \to 8 \to 10$
  i.e. bundle moved from 4 to 6 ...
- model with binary $x_{ilm} \in \{0, 1\}$
  $x_{ilm} = 1$
  $\Leftrightarrow$ node i has bundle l of cycle m



AMPL Model of Core Reload Operation

Exactly one bundle per node:

$$\sum_{l=1}^{L} \sum_{m=1}^{M} x_{ilm} = 1 \qquad \forall i \in I$$

AMPL model:

```
var x {I,L,M} binary ;
Bundle {i in I}:  sum{l in L, m in M} x[i,l,m] = 1 ;
```

- Multiple Choice: One of the most common uses of IP
- Full AMPL model `c-reload.mod` at `www.mcs.anl.gov/~leyffer/MacMINLP/`

Gas Transmission Problem (De Wolf and Smeers, 2000)

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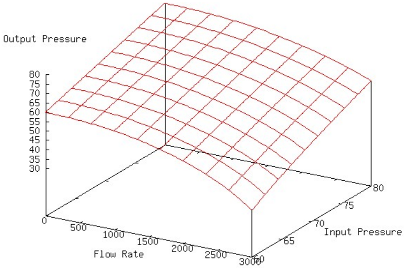
- Supply gas to all demand points in a network in a minimum cost fashion.
- Gas is pumped through the network with a series of compressors
- There are constraints on the pressure of the gas within the pipe

Pressure Loss is Nonlinear



- Assume horizontal pipes and steady state flows
- Pressure loss p across a pipe is related to the flow rate f as

$$p_{in}^2 - p_{out}^2 = \frac{1}{\Psi} \operatorname{sign}(f) f^2$$

- Ψ: "Friction Factor"

Gas Transmission: Problem Input

- Network (N, A). $A = A_p \cup A_a$
  - A_a: active arcs have compressor. Flow rate can increase on arc
  - A_p: passive arcs simply conserve flow rate
- $N_s \subseteq N$: set of supply nodes
- $c_i, i \in N_s$: Purchase cost of gas
- $\underline{s}_i, \overline{s}_i$: Lower and upper bounds on gas “supply” at node i
- $\underline{p}_i, \overline{p}_i$: Lower and upper bounds on gas pressure at node i

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- $s_i, i \in N$: supply at node i.
  - $s_i > 0 \Rightarrow$ gas added to the network at node i
  - $s_i < 0 \Rightarrow$ gas removed from the network at node i to meet demand
- $f_{ij}, (i, j) \in A$: flow along arc (i, j)
  - $f(i, j) > 0 \Rightarrow$ gas flows $i \rightarrow j$
  - $f(i, j) < 0 \Rightarrow$ gas flows $j \rightarrow i$

Gas Transmission Model

$$\min \sum_{j \in N_s} c_j s_j$$

subject to

$$\begin{aligned}
\sum_{j|(i,j)\in A} f_{ij} &= s_i & \forall i \in N \\
\text{sign}(f_{ij}) f_{ij}^2 - \Psi_{ij}(p_i^2 - p_j^2) &= 0 & \forall (i,j) \in A_p \\
\text{sign}(f_{ij}) f_{ij}^2 - \Psi_{ij}(p_i^2 - p_j^2) &\geq 0 & \forall (i,j) \in A_a \\
s_i &\in [\underline{s}_i, \overline{s}_i] & \forall i \in N \\
p_i &\in [\underline{p}_i, \overline{p}_i] & \forall i \in N \\
f_{ij} &\geq 0 & \forall (i,j) \in A_a
\end{aligned}$$

Your First Modeling Trick

- Don't include nonlinearities or nonconvexities unless necessary!
- Replace $p_i^2 \leftarrow \rho_i$

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f_{ij}^2 - \Psi_{ij}(\rho_i - \rho_j) &\geq 0 \qquad \forall (i,j) \in A_a \\
\rho_i &\in [\sqrt{\underline{p}_i}, \sqrt{\overline{p}_i}] \qquad \forall i \in N
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- This trick only works because
  1. p_i^2 terms appear only in the bound constraints
  2. Also $f_{ij} \geq 0 \ \forall (i,j) \in A_a$
- This model is nonconvex: $\text{sign}(f_{ij})f_{ij}^2$ is a nonconvex function
- Some solvers do not like sign

Dealing with sign(·): The NLP Way

- Use auxiliary binary variables to indicate direction of flow
- Let $|f_{ij}| \leq F \; \forall (i,j) \in A_p$

$$z_{ij} = \begin{cases} 1 & f_{ij} \geq 0 \\ 0 & f_{ij} \leq 0 \end{cases} \qquad \begin{aligned} f_{ij} &\geq -F(1 - z_{ij}) \\ f_{ij} &\leq F z_{ij} \end{aligned}$$

- Note that

$$\text{sign}(f_{ij}) = 2z_{ij} - 1$$

- Write constraint as

$$(2z_{ij} - 1)f_{ij}^2 - \Psi_{ij}(\rho_i - \rho_j) = 0.$$

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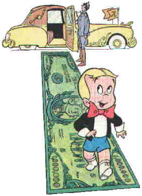
- Heuristic for finding a good starting solution, then a local optimization approach based on a piecewise-linear simplex method
- Another (similar) approach involves approximating the nonlinear function by piecewise linear segments, but searching for the globally optimal solution: Special Ordered Sets of Type 2
- If the "multidimensional" nonlinearity cannot be removed, resort to Special Ordered Sets of Type 3

Portfolio Management



- N: Universe of asset to purchase
- x_i: Amount of asset i to hold
- B: Budget

$$\min_{x \in \mathbb{R}_+^{|N|}} \left\{ u(x) \mid \sum_{i \in N} x_i = B \right\}$$

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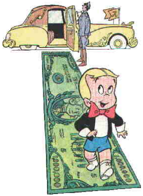

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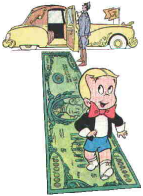
- Markowitz: $u(x) \stackrel{\text{def}}{=} -\alpha^T x + \lambda x^T Q x$
  - α: Expected returns
  - Q: Variance-covariance matrix of expected returns
  - λ: Risk aversion parameter



More Realistic Models

- $b \in \mathbb{R}^{|N|}$ of "benchmark" holdings
- Benchmark Tracking: $u(x) \stackrel{\text{def}}{=} (x - b)^T Q(x - b)$
  - Constraint on $\mathbb{E}$[Return]: $\alpha^T x \geq r$

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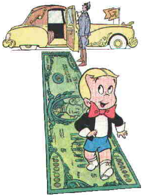

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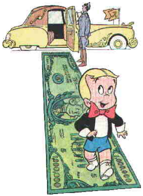
- Limit Names: $|i \in N \;:\; x_i > 0| \leq K$
  - Use binary indicator variables to model the implication $x_i > 0 \Rightarrow y_i = 1$
  - Implication modeled with variable upper bounds:

$$x_i \leq By_i \qquad \forall i \in N$$

  - $\sum_{i \in N} y_i \leq K$

Even More Models



- Min Holdings: $(x_i = 0) \vee (x_i \geq m)$
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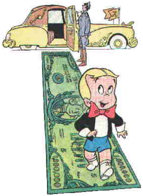

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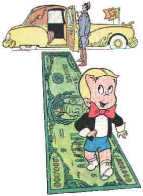
- Round Lots: $x_i \in \{kL_i, k = 1, 2, \ldots\}$
  - $x_i - z_i L_i = 0, z_i \in \mathbb{Z}_+ \; \forall i \in N$



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  - $x_i - z_i L_i = 0, z_i \in \mathbb{Z}_+ \ \forall i \in N$
- Vector h of initial holdings
- Transactions: $t_i = |x_i - h_i|$
- Turnover: $\sum_{i \in N} t_i \leq \Delta$
- Transaction Costs: $\sum_{i \in N} c_i t_i$ in objective
- Market Impact: $\sum_{i \in N} \gamma_i t_i^2$ in objective

Multiproduct Batch Plants (Kocis and Grossmann, 1988)

- M: Batch Processing Stages
- N: Different Products
- H: Horizon Time
- Q_i: Required quantity of product i
- t_{ij}: Processing time product i stage j
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- B_i: Batch size of product $i \in N$
- V_j: Stage j size: $V_j \geq S_{ij} B_i \; \forall i, j$
- N_j: Number of machines at stage j
- C_i: Longest stage time for product i: $C_i \geq t_{ij} / N_j \; \forall i, j$

Multiproduct Batch Plants

$$\min \sum_{j \in M} \alpha_j N_j V_j^{\beta_j}$$

s.t.

$$\begin{array}{rcll} V_j - S_{ij} B_i & \geq & 0 & \forall i \in N, \forall j \in M \\ C_i N_j & \geq & t_{ij} & \forall i \in N, \forall j \in M \\ \sum_{i \in N} \frac{Q_i}{B_i} C_i & \leq & H & \\ \text{Bound Constraints} & \text{on} & V_j, C_i, B_i, N_j & \\ N_j & \in & \mathbb{Z} & \forall j \in M \end{array}$$

Modeling Trick #2

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- Sometimes variable transformations work!

$$v_j = \ln(V_j), n_j = \ln(N_j), b_i = \ln(B_i), c_i = \ln C_i$$

Modeling Trick #2

- Horizon Time and Objective Function Nonconvex. :-(
- Sometimes variable transformations work!

$$v_j = \ln(V_j), n_j = \ln(N_j), b_i = \ln(B_i), c_i = \ln C_i$$

$$\min \sum_{j \in M} \alpha_j e^{N_j + \beta_j V_j}$$

$$\begin{array}{rcll} \text{s.t. } v_j - \ln(S_{ij}) b_i & \geq & 0 & \forall i \in N, \forall j \in M \\ c_i + n_j & \geq & \ln(\tau_{ij}) & \forall i \in N, \forall j \in M \\ \sum_{i \in N} Q_i e^{C_i - B_i} & \leq & H & \\ \text{(Transformed) Bound Constraints} & \text{on} & V_j, C_i, B_i & \end{array}$$

How to Handle the Integrality?

- But what to do about the integrality?

$$1 \leq N_j \leq \overline{N}_j \qquad \forall j \in M, N_j \in \mathbb{Z} \qquad \forall j \in M$$

- $n_j \in \{0, \ln(2), \ln(3), \ldots \ldots\}$

$$Y_{kj} = \begin{cases} 1 & n_j \text{ takes value } \ln(k) \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned} n_j - \sum_{k=1}^{K} \ln(k) Y_{kj} &= 0 \qquad \forall j \in M \\ \sum_{k=1}^{K} Y_{kj} &= 1 \qquad \forall j \in M \end{aligned}$$

- This model is available at `http://www-unix.mcs.anl.gov/~leyffer/macminlp/problems/batch.mod`

A Small Smattering of Other Applications

- Chemical Engineering Applications:
  - process synthesis (Kocis and Grossmann, 1988)
  - batch plant design (Grossmann and Sargent, 1979)
  - cyclic scheduling (Jain, V. and Grossmann, I.E., 1998)
  - design of distillation columns (Viswanathan and Grossmann, 1993)
  - pump configuration optimization (Westerlund, T., Pettersson, F. and Grossmann, I.E., 1994)
- Forestry/Paper
  - production (Westerlund, T., Isaksson, J. and Harjunkoski, I., 1995)
  - trimloss minimization (Harjunkoski, I., Westerlund, T., Pörn, R. and Skrifvars, H., 1998)
- Topology Optimization (Sigmund, 2001)

Part II

Classical Solution Methods

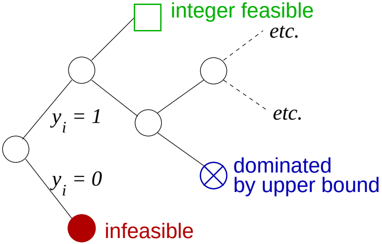
Classical Solution Methods for MINLP

1. Classical Branch-and-Bound
2. Outer Approximation & Benders Decomposition
3. Hybrid Methods
   - LP/NLP Based Branch-and-Bound
   - Integrating SQP with Branch-and-Bound

Branch-and-Bound

Solve relaxed NLP ($0 \leq y \leq 1$ continuous relaxation)
...solution value provides lower bound

- Branch on y_i non-integral
- Solve NLPs & branch until
  1. Node infeasible ... ●
  2. Node integer feasible ... □
     $\Rightarrow$ get upper bound (U)
  3. Lower bound $\geq U$ ... $\otimes$



Search until no unexplored nodes on tree

Variable Selection for Branch-and-Bound

Assume $y_i \in \{0,1\}$ for simplicity ...
$(\hat{x},\hat{y})$ fractional solution to parent node; $\hat{f} = f(\hat{x},\hat{y})$

1. **maximal fractional branching:** choose $\hat{y}_i$ closest to $\frac{1}{2}$

$$\max_i \{\min(1-\hat{y}_i, \hat{y}_i)\}$$

Variable Selection for Branch-and-Bound

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$$\max_i \{\min(1 - \hat{y}_i, \hat{y}_i)\}$$

2. **strong branching**: (approx) solve *all* NLP children:

$$f_i^{+/-} \leftarrow \left\{ \begin{array}{ll} \underset{x,y}{\text{minimize}} & f(x, y) \\ \text{subject to} & c(x, y) \leq 0 \\ & x \in X, \; y \in Y, \; y_i = 1/0 \end{array} \right.$$

branching variable y_i that changes objective the most:

$$\max_i \left\{\min(f_i^+, f_i^-)\right\}$$

Node Selection for Branch-and-Bound

Which node n on tree $\mathcal{T}$ should be solved next?

1. **depth-first search:** select deepest node in tree
   - minimizes number of NLP nodes stored
   - exploit warm-starts (MILP/MIQP only)

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Which node n on tree $\mathcal{T}$ should be solved next?

1. **depth-first search:** select deepest node in tree
   - minimizes number of NLP nodes stored
   - exploit warm-starts (MILP/MIQP only)
2. **best estimate:** choose node with best expected integer soln

$$\min_{n \in \mathcal{T}} \left\{ f_{p(n)} + \sum_{i: y_i \text{fractional}} \min\left\{ e_i^+(1 - y_i), e_i^- y_i \right\} \right\}$$

where $f_{p(n)}$ = value of parent node, $e_i^{+/-}$ = pseudo-costs
summing pseudo-cost estimates for all integers in subtree

Outer Approximation (Duran and Grossmann, 1986)

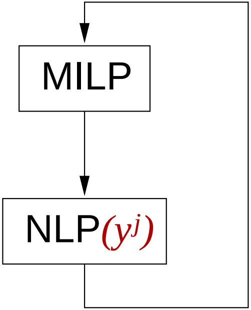
Motivation: avoid solving huge number of NLPs
- Exploit MILP/NLP solvers: decompose integer/nonlinear part

Key idea: reformulate MINLP as MILP (implicit)
- Solve alternating sequence of MILP & NLP

NLP subproblem y_j fixed:

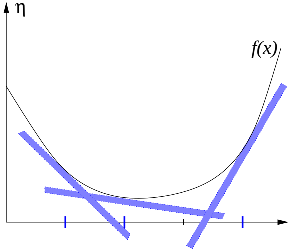
$$\text{NLP}(y_j) \begin{cases} \underset{x}{\text{minimize}} & f(x, y_j) \\ \text{subject to} & c(x, y_j) \leq 0 \\ & x \in X \end{cases}$$

Main Assumption: f, c are convex



Outer Approximation (Duran and Grossmann, 1986)

- let (x_j, y_j) solve NLP(y_j)
- linearize f, c about $(x_j, y_j) =: z_j$
- new objective variable $\eta \geq f(x, y)$
- MINLP $(P) \equiv$ MILP (M)

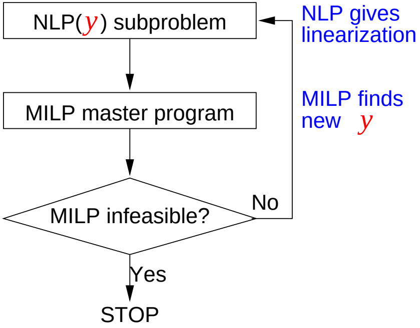


$$
(M) \begin{cases} \underset{z=(x,y),\eta}{\text{minimize}} & \eta \\ \text{subject to} & \eta \geq f_j + \nabla f_j^T (z - z_j) \quad \forall y_j \in Y \\ & 0 \geq c_j + \nabla c_j^T (z - z_j) \quad \forall y_j \in Y \\ & x \in X, \; y \in Y \text{ integer} \end{cases}
$$

SNAG: need all $y_j \in Y$ linearizations

Outer Approximation (Duran and Grossmann, 1986)

(M_k): lower bound (underestimate convex f, c)
NLP(y_j): upper bound U (fixed y_j)



$\Rightarrow$ stop, if lower bound $\geq$ upper bound

Outer Approximation & Benders Decomposition

Take OA cuts for $z_j := (x_j, y_j)$ ... wlog $X = \mathbb{R}^n$

$$\eta \geq f_j + \nabla f_j^T (z - z_j) \quad \& \quad 0 \geq c_j + \nabla c_j^T (z - z_j)$$

sum with $(1, \lambda_j)$ ... λ_j multipliers of NLP(y_j)

$$\eta \geq f_j + \lambda_j^T c_j + (\nabla f_j + \nabla c_j \lambda_j)^T (z - z_j)$$

KKT conditions of NLP(y_j) $\Rightarrow \quad \nabla_x f_j + \nabla_x c_j \lambda_j = 0$
... eliminate x components from valid inequality in y

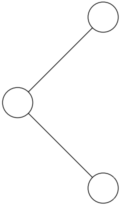
$$\Rightarrow \quad \eta \geq f_j + (\nabla_y f_j + \nabla_y c_j \lambda_j)^T (y - y_j)$$

NB: $\mu_j = \nabla_y f_j + \nabla_y c_j \lambda_j$ multiplier of $y = y_j$ in NLP(y_j)
References: (Geoffrion, 1972)

LP/NLP Based Branch-and-Bound

AIM: avoid re-solving MILP master (M)

- Consider MILP branch-and-bound



LP/NLP Based Branch-and-Bound

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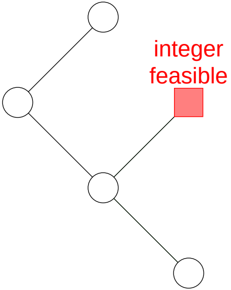
- Consider MILP branch-and-bound
- interrupt MILP, when y_j found
  $\Rightarrow$ solve NLP(y_j) get x_j



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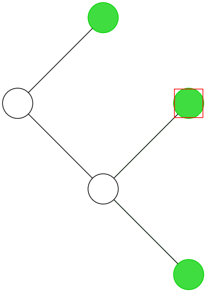
  $\Rightarrow$ add linearization to tree



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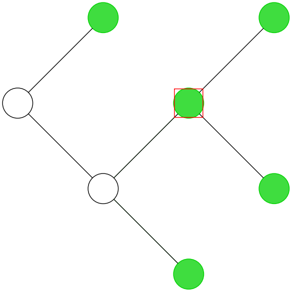
  $\Rightarrow$ add linearization to tree
- continue MILP tree-search



... until lower bound $\geq$ upper bound

LP/NLP Based Branch-and-Bound

- need access to MILP solver ... call back
  - exploit good MILP (branch-cut-price) solver
  - (Akrotirianakis et al., 2001) use Gomory cuts in tree-search

- preliminary results: order of magnitude faster than OA
  - same number of NLPs, but only one MILP
- similar ideas for Benders & Extended Cutting Plane methods
- recent implementation by CMU/IBM group

References: (Quesada and Grossmann, 1992)

Integrating SQP & Branch-and-Bound

AIM: Avoid solving NLP node to convergence.

Sequential Quadratic Programming (SQP)
$\rightarrow$ solve sequence (QP_k) at every node

$$
(QP_k) \begin{cases} \underset{d}{\text{minimize}} & f_k + \nabla f_k^T d + \frac{1}{2} d^T H_k d \\ \text{subject to} & c_k + \nabla c_k^T d \leq 0 \\ & x_k + d_x \in X \\ & y_k + d_y \in \hat{Y}. \end{cases}
$$

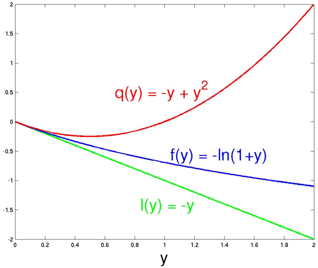
Early branching:
After QP step choose non-integral y_i^{k+1}, branch & continue SQP
References: (Borchers and Mitchell, 1994; Leyffer, 2001)

Integrating SQP & Branch-and-Bound

SNAG: (QP_k) not lower bound

$\Rightarrow$ no fathoming from upper bound

$$\begin{array}{ll} \underset{d}{\text{minimize}} & f_k + \nabla f_k^T d + \frac{1}{2} d^T H_k d \\ \text{subject to} & c_k + \nabla c_k^T d \leq 0 \\ & x_k + d_x \in X \\ & y_k + d_y \in \hat{Y}. \end{array}$$



NB: (QP_k) inconsistent and trust-region active $\Rightarrow$ do not fathom

Integrating SQP & Branch-and-Bound

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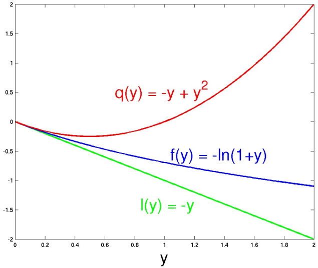


Remedy: Exploit OA underestimating property (Leyffer, 2001):

- add objective cut $f_k + \nabla f_k^T d \leq U - \epsilon$ to (QP_k)
- fathom node, if (QP_k) inconsistent

NB: (QP_k) inconsistent and trust-region active $\Rightarrow$ do not fathom

Comparison of Classical MINLP Techniques

Summary of numerical experience

1. Quadratic OA master: usually fewer iteration
   MIQP harder to solve
2. NLP branch-and-bound faster than OA
   ... depends on MIP solver
3. LP/NLP-based-BB order of magnitude faster than OA
   . . . also faster than B&B
4. Integrated SQP-B&B up to $3\times$ faster than B&B
   $\simeq$ number of QPs per node
5. ECP works well, if function/gradient evals expensive

Part III

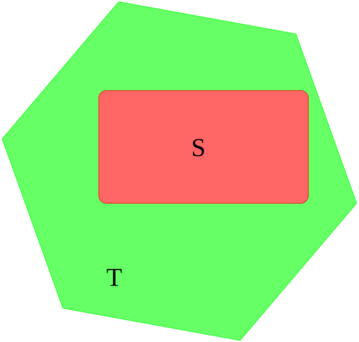
Modern Developments in MINLP

Modern Methods for MINLP

1. Formulations
   - Relaxations
   - Good formulations: big $M's$ and disaggregation
2. Cutting Planes
   - Cuts from relaxations and special structures
   - Cuts from integrality
3. Handling Nonconvexity
   - Envelopes
   - Methods

Relaxations

- $z(S) \stackrel{\text{def}}{=} \min_{x \in S} f(x)$
- $z(T) \stackrel{\text{def}}{=} \min_{x \in T} f(x)$



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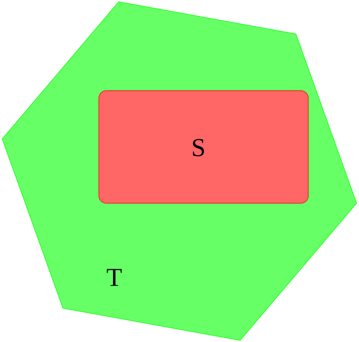

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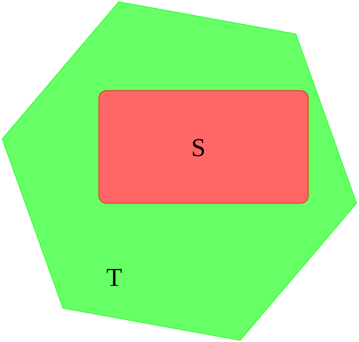

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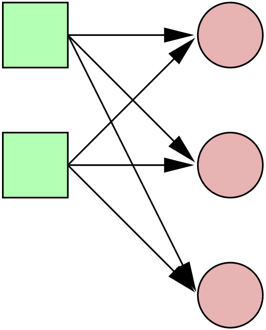
UFL: Uncapacitated Facility Location

- Facilities: J
- Customers: I



$$\min \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} f_{ij} y_{ij}$$

$$\sum_{j \in J} y_{ij} = 1 \quad \forall i \in I$$

$$\sum_{i \in I} y_{ij} \leq |I| x_j \quad \forall j \in J \qquad (1)$$

OR $y_{ij} \leq x_j \quad \forall i \in I, \; j \in J \qquad (2)$

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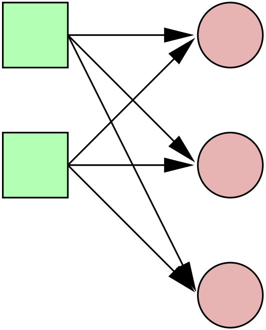

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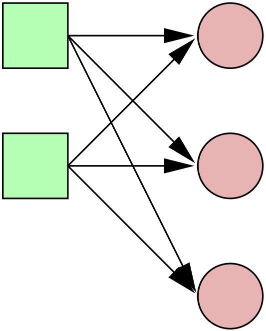

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- Which formulation is to be preferred?
- $I = J = 40$. Costs random.

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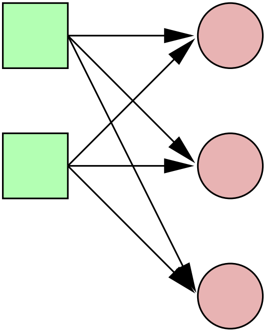

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- Which formulation is to be preferred?
- $I = J = 40$. Costs random.
  - Formulation 1. 53,121 seconds, optimal solution.
  - Formulation 2. 2 seconds, optimal solution.

Valid Inequalities

- Sometimes we can get a better formulation by dynamically improving it.

- An inequality $\pi^T x \leq \pi_0$ is a valid inequality for S if $\pi^T x \leq \pi_0 \; \forall x \in S$
- Alternatively: $\max_{x \in S}\{\pi^T x\} \leq \pi_0$

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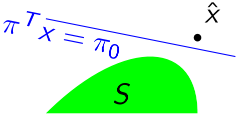
- **Thm:** (Hahn-Banach). Let $S \subset \mathbb{R}^n$ be a closed, convex set, and let $\hat{x} \notin S$. Then there exists $\pi \in \mathbb{R}^n$ such that

$$\pi^T \hat{x} > \max_{x \in S}\{\pi^T x\}$$



Nonlinear Branch-and-Cut

Consider MINLP

$$\begin{cases} \underset{x,y}{\text{minimize}} & f_x^T x + f_y^T y \\ \text{subject to} & c(x,y) \leq 0 \\ & y \in \{0,1\}^p, \; 0 \leq x \leq U \end{cases}$$

- Note the Linear objective
- This is WLOG:

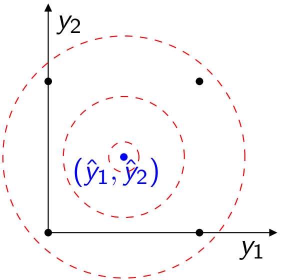
$$\min \; f(x,y) \qquad \Leftrightarrow \qquad \min \; \eta \;\; \text{s.t.} \;\; \eta \geq f(x,y)$$

It's Actually Important!

- We want to approximate the convex hull of integer solutions, but without a linear objective function, the solution to the relaxation might occur in the interior.
- No Separating Hyperplane! :-(

$$\min (y_1 - 1/2)^2 + (y_2 - 1/2)^2$$

$$\text{s.t. } y_1 \in \{0, 1\}, y_2 \in \{0, 1\}$$



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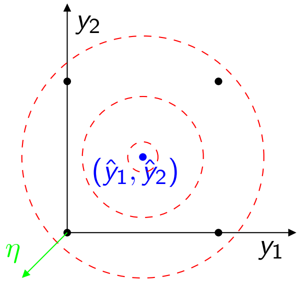
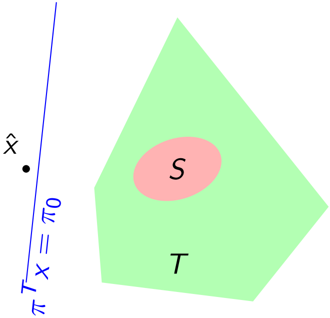


Valid Inequalities From Relaxations

- Idea: Inequalities valid for a relaxation are valid for original
- Generating valid inequalities for a relaxation is often easier.



- Separation Problem over T: Given $\hat{x}$, T find (π, π_0) such that $\pi^T \hat{x} > \pi_0$, $\pi^T x \leq \pi_0 \forall x \in T$

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- If the intersection of the relaxations is a good approximation to the true problem, then the inequalities will be quite useful.
- Crowder et al. (1983) is the seminal paper that shows this to be true for IP.
- MINLP: Single (linear) row relaxations are also valid $\Rightarrow$ same inequalities can also be used

Knapsack Covers

$$K = \{x \in \{0,1\}^n \mid a^T x \leq b\}$$

- A set $C \subseteq N$ is a cover if $\sum_{j \in C} a_j > b$
- A cover C is a minimal cover if $C \setminus j$ is not a cover $\forall j \in C$

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- A cover C is a minimal cover if $C \setminus j$ is not a cover $\forall j \in C$
- If $C \subseteq N$ is a cover, then the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

  is a valid inequality for S
- Sometimes (minimal) cover inequalities are facets of $\text{conv}(K)$

Other Substructures

- Single node flow: (Padberg et al., 1985)

$$S = \left\{ x \in \mathbb{R}_+^{|N|}, y \in \{0,1\}^{|N|} \mid \sum_{j \in N} x_j \leq b, x_j \leq u_j y_j \; \forall \; j \in N \right\}$$

- Knapsack with single continuous variable: (Marchand and Wolsey, 1999)

$$S = \left\{ x \in \mathbb{R}_+, y \in \{0,1\}^{|N|} \mid \sum_{j \in N} a_j y_j \leq b + x \right\}$$

- Set Packing: (Borndörfer and Weismantel, 2000)

$$S = \left\{ y \in \{0,1\}^{|N|} \mid Ay \leq e \right\}$$

$A \in \{0,1\}^{|M| \times |N|}, e = (1, 1, \ldots, 1)^T$

The Chvátal-Gomory Procedure

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- Let the columns of $A \in \mathbb{R}^{m \times n}$ be denoted by $\{a_1, a_2, \ldots a_n\}$
- $S = \{y \in \mathbb{Z}^n_+ \mid Ay \leq b\}$.
  1. Choose nonnegative multipliers $u \in \mathbb{R}^m_+$
  2. $u^T Ay \leq u^T b$ is a valid inequality ($\sum_{j \in N} u^T a_j y_j \leq u^T b$).
  3. $\sum_{j \in N} \lfloor u^T a_j \rfloor y_j \leq u^T b$ (Since $y \geq 0$).
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    4. $\sum_{j \in N} \lfloor u^T a_j \rfloor y_j \leq \lfloor u^T b \rfloor$ is valid for S since $\lfloor u^T a_j \rfloor y_j$ is an integer
- Simply Amazing: This simple procedure suffices to generate every valid inequality for an integer program

Extension to MINLP (Çezik and Iyengar, 2005)

- This simple idea also extends to mixed 0-1 conic programming

$$\left\{\begin{array}{ll} \underset{z \stackrel{\text{def}}{=} (x,y)}{\text{minimize}} & f^T z \\ \text{subject to} & Az \succeq_{\mathcal{K}} b \\ & y \in \{0,1\}^p, \; 0 \le x \le U \end{array}\right.$$

- $\mathcal{K}$: Homogeneous, self-dual, proper, convex cone
- $x \succeq_{\mathcal{K}} y \Leftrightarrow (x - y) \in \mathcal{K}$

Gomory On Cones (Çezik and Iyengar, 2005)

- LP: $\mathcal{K}_l = \mathbb{R}^n_+$
- SOCP: $\mathcal{K}_q = \{(x_0, \bar{x}) \mid x_0 \geq \|\bar{x}\|\}$
- SDP: $\mathcal{K}_s = \{x = \text{vec}(X) \mid X = X^T, X \text{ p.s.d}\}$

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$$Az \succeq_{\mathcal{K}} b \quad \Leftrightarrow \quad u^T Az \geq u^T b \; \forall u \succeq_{\mathcal{K}^*} 0$$

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- Many classes of nonlinear inequalities can be represented as

$$Ax \succeq_{\mathcal{K}_q} b \text{ or } Ax \succeq_{\mathcal{K}_s} b$$

Using Gomory Cuts in MINLP (Akrotirianakis et al., 2001)

- LP/NLP Based Branch-and-Bound solves MILP instances:

$$\begin{cases} \underset{z \stackrel{\text{def}}{=} (x,y),\eta}{\text{minimize}} & \eta \\ \text{subject to} & \eta \geq f_j + \nabla f_j^T (z - z_j) \quad \forall y_j \in Y^k \\ & 0 \geq c_j + \nabla c_j^T (z - z_j) \quad \forall y_j \in Y^k \\ & x \in X, \; y \in Y \text{ integer} \end{cases}$$

Using Gomory Cuts in MINLP (Akrotirianakis et al., 2001)

- LP/NLP Based Branch-and-Bound solves MILP instances:

$$\left\{ \begin{array}{lll} \underset{z \stackrel{\mathrm{def}}{=} (x,y), \eta}{\text{minimize}} & \eta & \\ \text{subject to} & \eta \geq f_j + \nabla f_j^T (z - z_j) & \forall y_j \in Y^k \\ & 0 \geq c_j + \nabla c_j^T (z - z_j) & \forall y_j \in Y^k \\ & x \in X, \; y \in Y \text{ integer} & \end{array} \right.$$

- Create Gomory mixed integer cuts from

$$\begin{array}{rcl} \eta & \geq & f_j + \nabla f_j^T (z - z_j) \\ 0 & \geq & c_j + \nabla c_j^T (z - z_j) \end{array}$$

- Akrotirianakis et al. (2001) shows modest improvements

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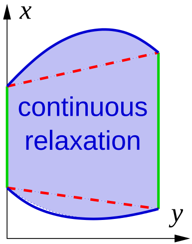
- Akrotirianakis et al. (2001) shows modest improvements
- Research Question: Other cut classes?
- Research Question: Exploit "outer approximation" property?

Disjunctive Cuts for MINLP (Stubbs and Mehrotra, 1999)

Extension of Disjunctive Cuts for MILP: (Balas, 1979; Balas et al., 1993)

Continuous relaxation ($z \stackrel{\text{def}}{=} (x, y)$)

- $C \stackrel{\text{def}}{=} \{z | c(z) \leq 0,\ 0 \leq y \leq 1,\ 0 \leq x \leq U\}$



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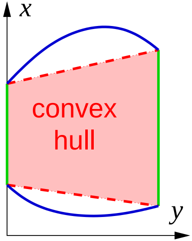
- $\mathcal{C} \stackrel{\text{def}}{=} \text{conv}(\{x \in C \mid y \in \{0,1\}^p\})$



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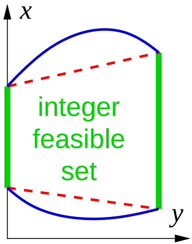
- $\mathcal{C} \stackrel{\text{def}}{=} \text{conv}(\{x \in C \mid y \in \{0,1\}^p\})$
- $C_j^{0/1} \stackrel{\text{def}}{=} \{z \in C | y_j = 0/1\}$



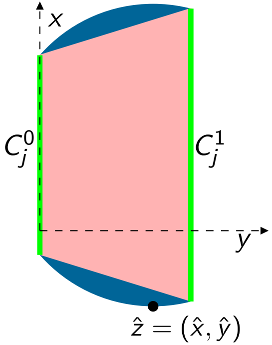
$$\text{let } \mathcal{M}_j(C) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} z = \lambda_0 u_0 + \lambda_1 u_1 \\ \lambda_0 + \lambda_1 = 1,\ \lambda_0, \lambda_1 \geq 0 \\ u_0 \in C_j^0,\ u_1 \in C_j^1 \end{array} \right\}$$

$\Rightarrow \mathcal{P}_j(C) :=$ projection of $\mathcal{M}_j(C)$ onto z

$\Rightarrow \mathcal{P}_j(C) = \text{conv}\,(C \cap y_j \in \{0,1\})$ and $\mathcal{P}_{1\ldots p}(C) = \mathcal{C}$

Disjunctive Cuts: Example

$$\underset{x,y}{\text{minimize}} \left\{ x \mid (x-1/2)^2 + (y-3/4)^2 \leq 1, -2 \leq x \leq 2, y \in \{0,1\} \right\}$$

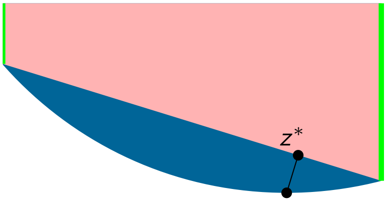


Given $\hat{z}$ with $\hat{y}_j \notin \{0,1\}$ find separating hyperplane

$$\Rightarrow \begin{cases} \underset{z}{\text{minimize}} & \|z - \hat{z}\| \\ \text{subject to} & z \in \mathcal{P}_j(C) \end{cases}$$

Disjunctive Cuts Example

$$z^* \stackrel{\text{def}}{=} \arg\min \|z - \hat{z}\|$$



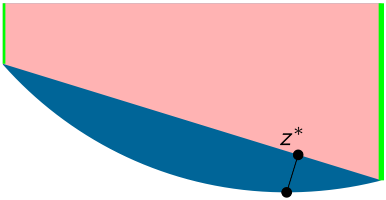
$\hat{z} = (\hat{x}, \hat{y})$

$$\begin{array}{rcl} \text{s.t.}\ \lambda_0 u_0 + \lambda_1 u_1 & = & z \\ \lambda_0 + \lambda_1 & = & 1 \\ \begin{pmatrix} -0.16 \\ 0 \end{pmatrix} \leq u_0 & \leq & \begin{pmatrix} 0.66 \\ 1 \end{pmatrix} \\ \begin{pmatrix} -0.47 \\ 0 \end{pmatrix} \leq u_1 & \leq & \begin{pmatrix} 1.47 \\ 1 \end{pmatrix} \\ \lambda_0, \lambda_1 & \geq & 0 \end{array}$$

Disjunctive Cuts Example

$$z^* \stackrel{\text{def}}{=} \arg\min \|z - \hat{z}\|_2^2$$

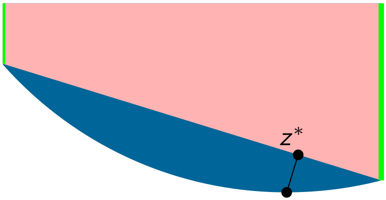
$$\begin{aligned} \text{s.t. } \lambda_0 u_0 + \lambda_1 u_1 &= z \\ \lambda_0 + \lambda_1 &= 1 \\ \begin{pmatrix} -0.16 \\ 0 \end{pmatrix} \leq u_0 &\leq \begin{pmatrix} 0.66 \\ 1 \end{pmatrix} \\ \begin{pmatrix} -0.47 \\ 0 \end{pmatrix} \leq u_1 &\leq \begin{pmatrix} 1.47 \\ 1 \end{pmatrix} \\ \lambda_0, \lambda_1 &\geq 0 \end{aligned}$$



$\hat{z} = (\hat{x}, \hat{y})$

Disjunctive Cuts Example

$$z^* \stackrel{\text{def}}{=} \arg\min \|z - \hat{z}\|_\infty$$



$\hat{z} = (\hat{x}, \hat{y})$

$$\begin{aligned}
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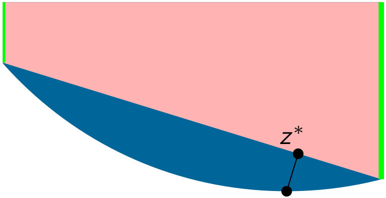
\end{aligned}$$

Disjunctive Cuts Example



$\hat{z} = (\hat{x}, \hat{y})$

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NONCONVEX

What to do? (Stubbs and Mehrotra, 1999)

- Look at the perspective of $c(z)$

$$\mathcal{P}(c(\tilde{z}), \mu) = \mu c(\tilde{z}/\mu)$$

- Think of $\tilde{z} = \mu z$

What to do? (Stubbs and Mehrotra, 1999)

- Look at the perspective of $c(z)$

$$\mathcal{P}(c(\tilde{z}), \mu) = \mu c(\tilde{z}/\mu)$$

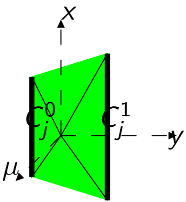
- Think of $\tilde{z} = \mu z$
- Perspective gives a convex reformulation of $\mathcal{M}_j(C)$: $\mathcal{M}_j(\tilde{C})$, where

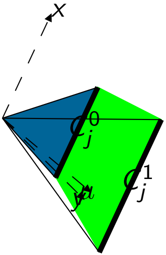
$$\tilde{C} := \left\{ (z, \mu) \left| \begin{array}{l} \mu c_i(z/\mu) \leq 0 \\ 0 \leq \mu \leq 1 \\ 0 \leq x \leq \mu U, \; 0 \leq y \leq \mu \end{array} \right. \right\}$$

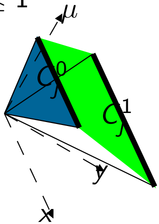
- $c(0/0) = 0 \Rightarrow$ convex representation

Disjunctive Cuts Example

$$\tilde{C} = \left\{ \begin{pmatrix} x \\ y \\ \mu \end{pmatrix} \middle| \begin{array}{c} \mu \left[(x/\mu - 1/2)^2 + (y/\mu - 3/4)^2 - 1 \right] \leq 0 \\ -2\mu \leq x \leq 2\mu \\ 0 \leq y \leq \mu \\ 0 \leq \mu \leq 1 \end{array} \right\}$$







Example, cont.

$$\tilde{C}_j^0 = \{(z, \mu) \mid y_j = 0\} \quad \tilde{C}_j^1 = \{(z, \mu) \mid y_j = \mu\}$$

Example, cont.

$$\tilde{C}_j^0 = \{(z,\mu) \mid y_j = 0\} \quad \tilde{C}_j^1 = \{(z,\mu) \mid y_j = \mu\}$$

- Take $v_0 \leftarrow \mu_0 u_0 \;\; v_1 \leftarrow \mu_1 u_1$

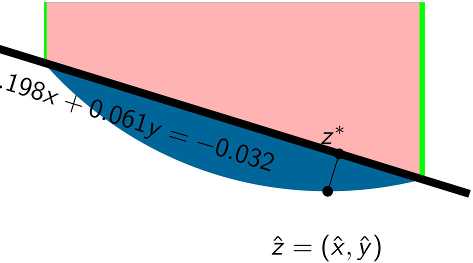
$$\begin{array}{rcl} \min \|z - \hat{z}\| & & \\ \text{s.t.} \;\; v_0 + v_1 & = & z \\ \mu_0 + \mu_1 & = & 1 \\ (v_0, \mu_0) & \in & \tilde{C}_j^0 \\ (v_1, \mu_1) & \in & \tilde{C}_j^1 \\ \mu_0, \mu_1 & \geq & 0 \end{array}$$

Solution to example:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} -0.401 \\ 0.780 \end{pmatrix}$$

- separating hyperplane: $\psi^T(z - \hat{z})$, where $\psi \in \partial\|z - \hat{z}\|$

Example, Cont.



$$\psi = \begin{pmatrix} 2x^* + 0.5 \\ 2y^* - 0.75 \end{pmatrix}$$

$$0.198x + 0.061y \geq -0.032$$

Nonlinear Branch-and-Cut (Stubbs and Mehrotra, 1999)

- Can do this at all nodes of the branch-and-bound tree
- Generalize disjunctive approach from MILP
  - solve one convex NLP per cut
- Generalizes Sherali and Adams (1990) and Lovász and Schrijver (1991)
  - tighten cuts by adding semi-definite constraint
- Stubbs and Mehrohtra (2002) also show how to generate convex quadratic inequalities, but computational results are not that promising

Generalized Disjunctive Programming (Raman and Grossmann, 1994; Lee and Grossmann, 2000)

Consider disjunctive NLP

$$\begin{cases} \underset{x,Y}{\text{minimize}} & \sum f_i \; + \; f(x) \\ \text{subject to} & \begin{bmatrix} Y_i \\ c_i(x) \leq 0 \\ f_i = \gamma_i \end{bmatrix} \bigvee \begin{bmatrix} \neg Y_i \\ B_i x = 0 \\ f_i = 0 \end{bmatrix} \forall i \in I \\ & 0 \leq x \leq U, \; \Omega(Y) = \text{true}, \; Y \in \{\text{true}, \text{false}\}^p \end{cases}$$

Application: process synthesis

- Y_i represents presence/absence of units
- $B_i x = 0$ eliminates variables if unit absent

Exploit disjunctive structure

- special branching ... OA/GBD algorithms

Generalized Disjunctive Programming (Raman and Grossmann, 1994; Lee and Grossmann, 2000)

Consider disjunctive NLP

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Big-M formulation (notoriously bad), $M > 0$:

$$c_i(x) \le M(1 - y_i)$$
$$-My_i \le B_i x \le My_i$$
$$f_i = y_i \gamma_i \qquad \Omega(Y) \text{ converted to linear inequalities}$$

Generalized Disjunctive Programming (Raman and Grossmann, 1994; Lee and Grossmann, 2000)

Consider disjunctive NLP

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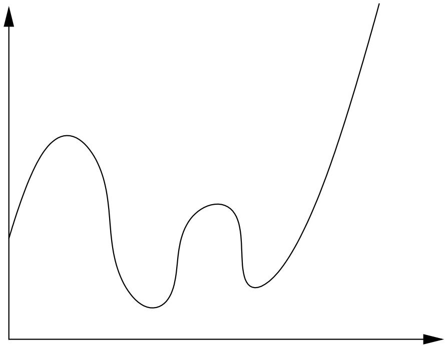
convex hull representation ...

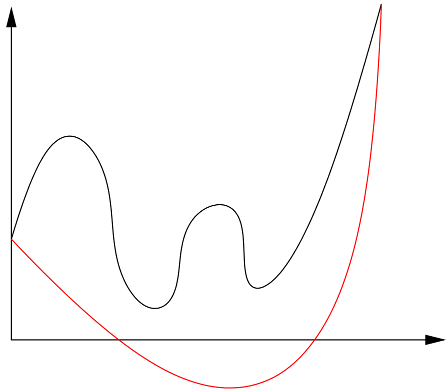
$$\begin{array}{lll} x = v_{i1} + v_{i0}, & \lambda_{i1} + \lambda_{i0} = 1 & \\ \lambda_{i1} c_i(v_{i1}/\lambda_{i1}) \le 0, & B_i v_{i0} = 0 & \\ 0 \le v_{ij} \le \lambda_{ij} U, & 0 \le \lambda_{ij} \le 1, & f_i = \lambda_{i1}\gamma_i \end{array}$$

Dealing with Nonconvexities



- Functional nonconvexity causes serious problems.
    - Branch and bound must have true lower bound (global solution)

Dealing with Nonconvexities



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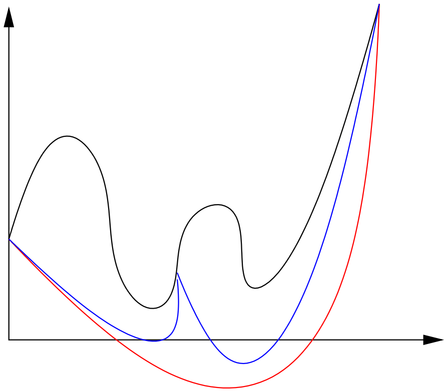
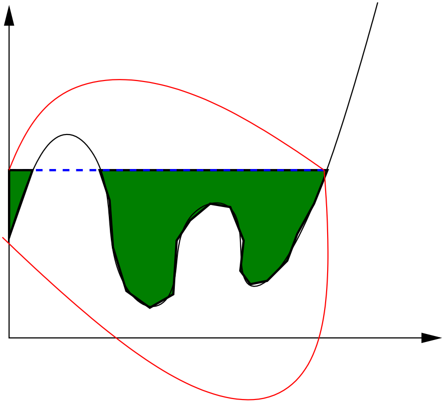
Dealing with Nonconvexities



- Functional nonconvexity causes serious problems.
    - Branch and bound must have true lower bound (global solution)
- Underestimate nonconvex functions. Solve relaxation. Provides lower bound.
- If relaxation is not exact, then branch

Dealing with Nonconvex Constraints

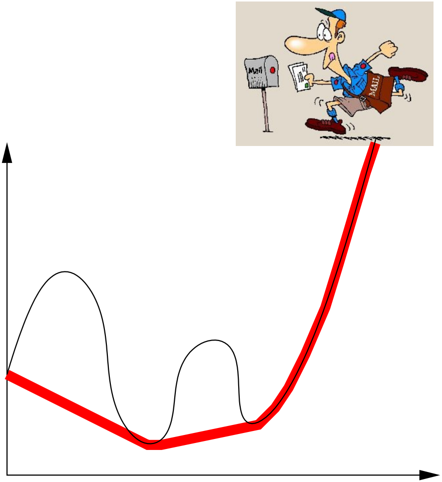


- If nonconvexity in constraints, may need to overestimate and underestimate the function to get a convex region

Envelopes

$$f : \Omega \to \mathbb{R}$$

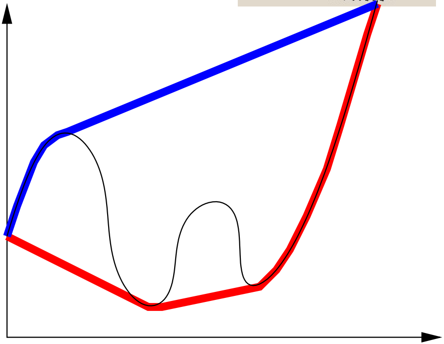
- Convex Envelope ($\text{vex}_{\Omega}(f)$): Pointwise supremum of convex underestimators of f over Ω.



Envelopes

$$f : \Omega \to \mathbb{R}$$

- Convex Envelope ($\text{vex}_\Omega(f)$): Pointwise supremum of convex underestimators of f over Ω.
- Concave Envelope ($\text{cav}_\Omega(f)$): Pointwise infimum of concave overestimators of f over Ω.



Branch-and-Bound Global Optimization Methods

- Under/Overestimate “simple” parts of (Factorable) Functions individually
  - Bilinear Terms
  - Trilinear Terms
  - Fractional Terms
  - Univariate convex/concave terms

Branch-and-Bound Global Optimization Methods

- Under/Overestimate "simple" parts of (Factorable) Functions individually
  - Bilinear Terms
  - Trilinear Terms
  - Fractional Terms
  - Univariate convex/concave terms
- General nonconvex functions $f(x)$ can be underestimated over a region $[l, u]$ "overpowering" the function with a quadratic function that is ≤ 0 on the region of interest

$$\mathcal{L}(x) = f(x) + \sum_{i=1}^{n} \alpha_i (l_i - x_i)(u_i - x_i)$$

Refs: (McCormick, 1976; Adjiman et al., 1998; Tawarmalani and Sahinidis, 2002)

Bilinear Terms

The convex and concave envelopes of the bilinear function xy over a rectangular region

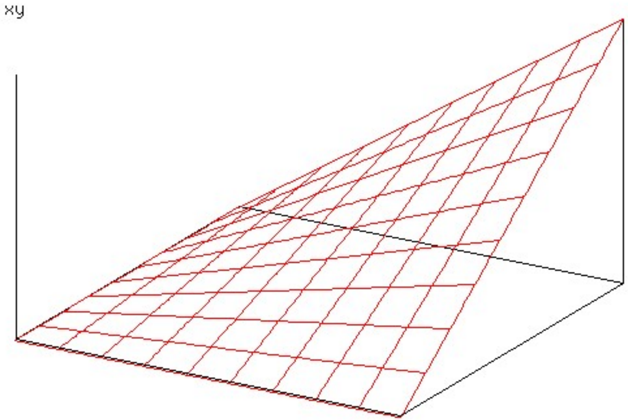
$$R \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{R}^2 \mid l_x \le x \le u_x,\ \ l_y \le y \le u_y\}$$

are given by the expressions

$$\begin{aligned} \text{vexxy}_R(x,y) &= \max\{l_y x + l_x y - l_x l_y,\, u_y x + u_x y - u_x u_y\} \\ \text{cavxy}_R(x,y) &= \min\{u_y x + l_x y - l_x u_y,\, l_y x + u_x y - u_x l_y\} \end{aligned}$$

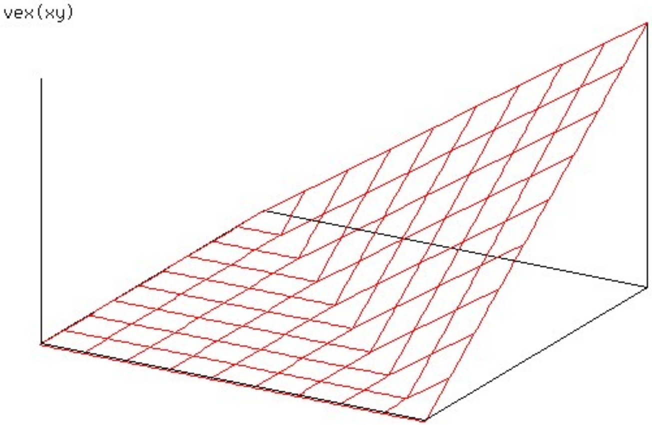
Worth 1000 Words?

xy



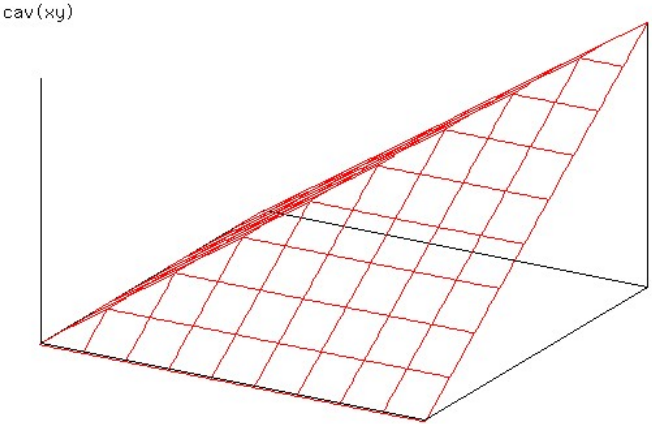
Worth 1000 Words?

$$\text{vex}_R(xy)$$



Worth 1000 Words?

$\text{cav}_R(xy)$



Summary

- MINLP: Good relaxations are important
- Relaxations can be improved
  - Statically: Better formulation/preprocessing
  - Dynamically: Cutting planes
- Nonconvex MINLP:
  - Methods exist, again based on relaxations
- Tight relaxations is an active area of research
- Lots of empirical questions remain

Part IV

Implementation and Software

Implementation and Software for MINLP

1. Special Ordered Sets
2. Implementation & Software Issues

Special Ordered Sets of Type 1

SOS1: $\sum \lambda_i = 1$ & at most one λ_i is nonzero

Example 1: $d \in \{d_1, \dots, d_p\}$ discrete diameters

$\Leftrightarrow d = \sum \lambda_i d_i$ and $\{\lambda_1, \dots, \lambda_p\}$ is SOS1

$\Leftrightarrow d = \sum \lambda_i d_i$ and $\sum \lambda_i = 1$ and $\lambda_i \in \{0, 1\}$

... d is convex combination with coefficients λ_i

Example 2: nonlinear function $c(y)$ of single integer

$\Leftrightarrow y = \sum i \lambda_i$ and $c = \sum c(i) \lambda_i$ and $\{\lambda_1, \dots, \lambda_p\}$ is SOS1

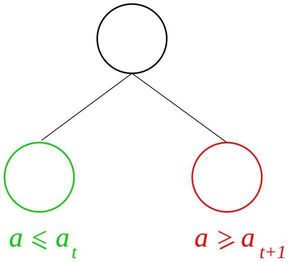
References: (Beale, 1979; Nemhauser, G.L. and Wolsey, L.A., 1988; Williams, 1993) ...

Special Ordered Sets of Type 1

SOS1: $\sum \lambda_i = 1$ & at most one λ_i is nonzero

Branching on SOS1

1. reference row $a_1 < \ldots < a_p$
   e.g. diameters
2. fractionality: $a := \sum a_i \lambda_i$

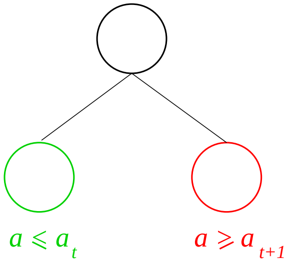


Special Ordered Sets of Type 1

SOS1: $\sum \lambda_i = 1$ & at most one λ_i is nonzero

Branching on SOS1

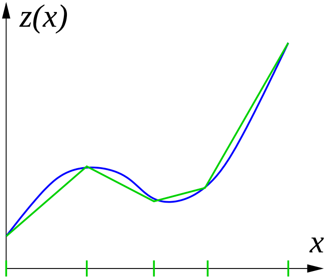
1. reference row $a_1 < \ldots < a_p$
   e.g. diameters
2. fractionality: $a := \sum a_i \lambda_i$
3. find t : $a_t < a \leq a_{t+1}$
4. branch: $\{\lambda_{t+1}, \ldots, \lambda_p\} = 0$
   or $\{\lambda_1, \ldots, \lambda_t\} = 0$



Special Ordered Sets of Type 2

SOS2: $\sum \lambda_i = 1$ & at most two adjacent λ_i nonzero

Example: Approximation of nonlinear function $z = z(x)$

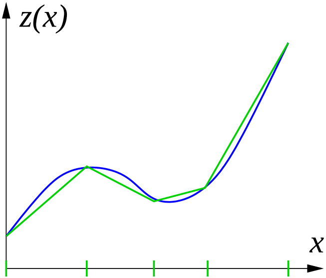


- breakpoints $x_1 < \ldots < x_p$
- function values $z_i = z(x_i)$
- piece-wise linear

Special Ordered Sets of Type 2

SOS2: $\sum \lambda_i = 1$ & at most two adjacent λ_i nonzero

Example: Approximation of nonlinear function $z = z(x)$



- breakpoints $x_1 < \ldots < x_p$
- function values $z_i = z(x_i)$
- piece-wise linear
- $x = \sum \lambda_i x_i$
- $z = \sum \lambda_i z_i$
- $\{\lambda_1, \ldots, \lambda_p\}$ is SOS2

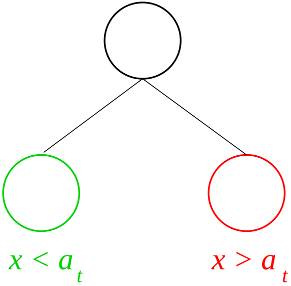
. . . convex combination of two breakpoints . . .

Special Ordered Sets of Type 2

SOS2: $\sum \lambda_i = 1$ & at most two adjacent λ_i nonzero

Branching on SOS2

1. reference row $a_1 < \ldots < a_p$
   e.g. $a_i = x_i$
2. fractionality: $a := \sum a_i \lambda_i$



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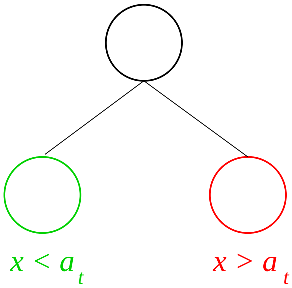
Branching on SOS2

1. reference row $a_1 < \ldots < a_p$
   e.g. $a_i = x_i$
2. fractionality: $a := \sum a_i \lambda_i$
3. find t : $a_t < a \leq a_{t+1}$
4. branch: $\{\lambda_{t+1}, \ldots, \lambda_p\} = 0$
   or $\{\lambda_1, \ldots, \lambda_{t-1}\}$

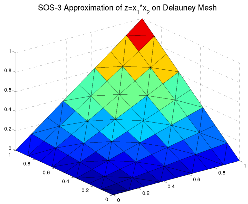


Special Ordered Sets of Type 3

Example: Approximation of 2D function $u = g(v, w)$

Triangularization of $[v_L, v_U] \times [w_L, w_U]$ domain

1. $v_L = v_1 < \ldots < v_k = v_U$
2. $w_L = w_1 < \ldots < w_l = w_U$
3. function $u_{ij} := g(v_i, w_j)$
4. λ_{ij} weight of vertex (i, j)



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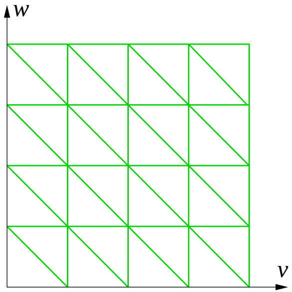
- $v = \sum \lambda_{ij} v_i$
- $w = \sum \lambda_{ij} w_j$
- $u = \sum \lambda_{ij} u_{ij}$



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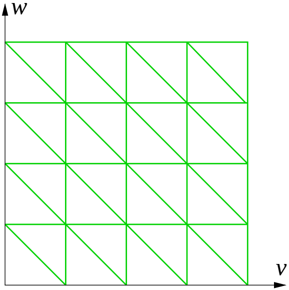


- $v = \sum \lambda_{ij} v_i$
- $w = \sum \lambda_{ij} w_j$
- $u = \sum \lambda_{ij} u_{ij}$

$1 = \sum \lambda_{ij}$ is SOS3 ...

Special Ordered Sets of Type 3

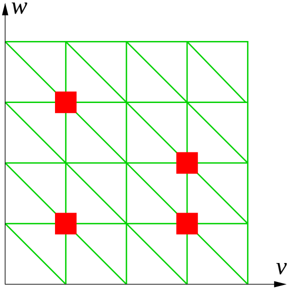
SOS3: $\sum \lambda_{ij} = 1$ & set condition holds

1. $v = \sum \lambda_{ij} v_i$ ... convex combinations
2. $w = \sum \lambda_{ij} w_j$
3. $u = \sum \lambda_{ij} u_{ij}$

$\{\lambda_{11}, \ldots, \lambda_{kl}\}$ satisfies set condition

$\Leftrightarrow\ \exists$ trangle $\Delta : \{(i,j) : \lambda_{ij} > 0\} \subset \Delta$

i.e. nonzeros in single triangle Δ

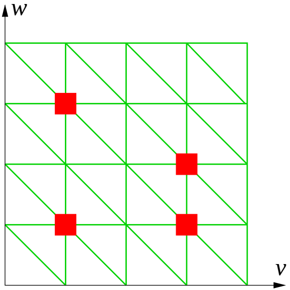


violates set condn

Branching on SOS3

λ violates set condition

- compute centers:
  $\hat{v} = \sum \lambda_{ij} v_i$ &
  $\hat{w} = \sum \lambda_{ij} w_i$
- find s, t such that
  $v_s \leq \hat{v} < v_{s+1}$ &
  $w_s \leq \hat{w} < w_{s+1}$
- branch on v or w



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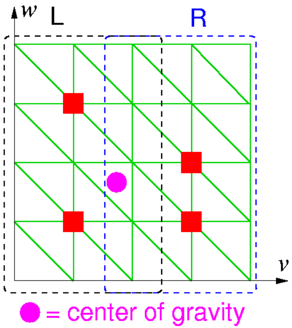

vertical branching: $\sum_L \lambda_{ij} = 1$ $\sum_R \lambda_{ij} = 1$

Branching on SOS3

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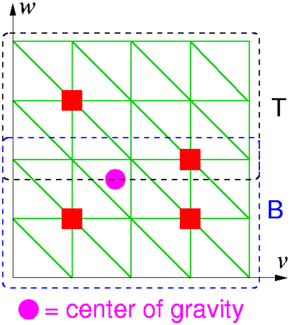

- compute centers:
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  $v_s \leq \hat{v} < v_{s+1}$ &
  $w_s \leq \hat{w} < w_{s+1}$
- branch on v or w

horizontal branching: $\sum_T \lambda_{ij} = 1$ $\sum_B \lambda_{ij} = 1$

Extension to SOS-k

Example: electricity transmission network:

$$c(x) = 4x_1 - x_2^2 - 0.2 \cdot x_2 x_4 \sin(x_3)$$

(Martin et al., 2005) extend SOS3 to SOSk models for any k
$\Rightarrow$ function with p variables on N grid needs N^p λ's

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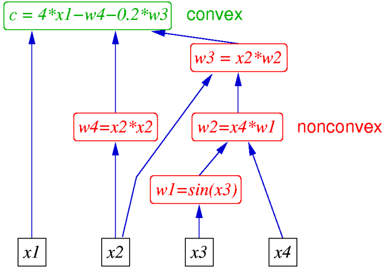
Alternative (Gatzke, 2005):

- exploit computational graph
  $\simeq$ automatic differentiation



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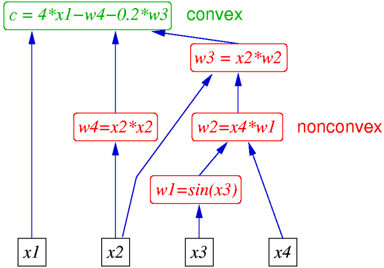

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Alternative (Gatzke, 2005):

- exploit computational graph $\simeq$ automatic differentiation
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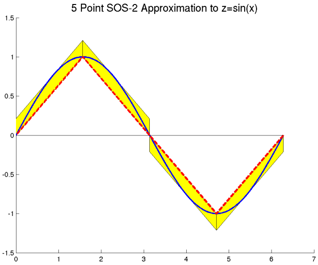
- piece-wise polyhedral approx.



Software for MINLP

- **Outer Approximation**: DICOPT++ (& AIMMS)
  NLP solvers: CONOPT, MINOS, SNOPT
  MILP solvers: CPLEX, OSL2
- **Branch-and-Bound Solvers**: SBB & MINLP
  NLP solvers: CONOPT, MINOS, SNOPT & FilterSQP
  variable & node selection; SOS1 & SOS2 support
- **Global MINLP**: BARON & MINOPT underestimators & branching
  CPLEX, MINOS, SNOPT, OSL
- **Online Tools**: MINLP World, MacMINLP & NEOS MINLP World
  `www.gamsworld.org/minlp/`
  NEOS server `www-neos.mcs.anl.gov/`

COIN-OR

http://www.coin-or.org

- COmputational INfrastructure for Operations Research
- A library of (interoperable) software tools for optimization
- A development platform for open source projects in the OR community
- Possibly Relevant Modules:
  - OSI: Open Solver Interface
  - CGL: Cut Generation Library
  - CLP: Coin Linear Programming Toolkit
  - CBC: Coin Branch and Cut
  - IPOPT: Interior Point OPTimizer for NLP
  - NLPAPI: NonLinear Programming API

MINLP with COIN-OR

New implementation of LP/NLP based BB

- MIP branch-and-cut: CBC & CGL
- NLPs: IPOPT interior point ... OK for NLP(y_i)
- New hybrid method:
  - solve more NLPs at non-integer y_i
    $\Rightarrow$ better outer approximation
  - allow complete MIP at some nodes
    $\Rightarrow$ generate new integer assignment

  ... faster than DICOPT++, SBB
- simplifies to OA and BB at extremes ... less efficient

... see Bonami et al. (2005) ... coming in 2006.

Conclusions

MINLP rich modeling paradigm
◦ most popular solver on NEOS

Algorithms for MINLP:
◦ Branch-and-bound (branch-and-cut)
◦ Outer approximation et al.

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"MINLP solvers lag 15 years behind MIP solvers"

$\Rightarrow$ many research opportunities!!!

Part V

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