Life-Cycle Portfolio Choice, the Wealth Distribution and Asset Prices

Felix Kubler[1] Karl Schmedders[2]

[1]Dept. of Banking and Finance, Univ. of Zurich and Swiss Finance Institute

[2]Dept. of Business Administration, Univ. of Zurich and Swiss Finance Institute

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Foundation of Many Asset Pricing Puzzles

Changes in observable fundamental parameters (labor income and dividends) cannot explain observed asset prices within

Want additional state variables as add'l features

- habit
- wealth distribution

But simple model enhancements usually yield little progress

- incomplete markets
- heterogeneous agents (with identical beliefs)
- borrowing constraints

Wealth Distribution

Huffman (1987)

- stylized stochastic OLG model
- asset prices depend on wealth distribution

Rios-Rull (1996)

- calibrated life-cycle model with complete markets and identical beliefs
- essentially same results as representative-agent models

Krusell and Smith (1998)

- stochastic growth model with ex ante identical agents and incomplete markets
- mean of wealth distribution suffices to describe equilibrium

Insufficient variability in the wealth distribution

Summary of the Paper

Canonical and parsimonious stochastic OLG model with dynamically complete markets and heterogeneous beliefs

Mechanism:

small differences in belief

$\Longrightarrow$

large movements in the wealth distribution

$\Longrightarrow$

substantial asset price volatility

Arrow-Debreu model delivering substantial volatility

Other Related Literature

Critique of common prior assumption (Morris, 1995)

Differences in beliefs and asset prices (Harrison and Kreps, 1978)

Market selection hypothesis and the survival and price impact of noise traders (DeLong et al., 1990, Sandroni, 2000, Blume and Easley, 2006, Kogan et al., 2006)

Outline

Stochastic OLG Economies

Time indexed by $t = 0, 1, 2, \ldots$.

Markov chain of exogenous shocks, $s_t \in \mathcal{S} = \{1, 2, \ldots, S\}$

"True" law of motion: $S \times S$ transition matrix Π

History of shocks $s^t = (s_0, s_1, \ldots, s_t)$, called date-event

Each period H agents are born, live for N periods

Agent identified by date-event s^t at birth and type $h = 1, 2, \ldots, H$

Stochastic OLG Economies cont'd

Single perishable consumption good

Individual endowments depend on shock, age a and type h

$$e^{s^t,h}(s^{t+a-1}) = e^{a,h}(s_{t+a-1})$$

Time-separable expected utility function

$$U^{s^t,h}(c) = \log\left(c(s^t)\right) + \sum_{a=1}^{N-1} \delta^a \sum_{s^{t+a} \succeq s^t} \pi^{a,h}(s^{t+a}|s^t) \log\left(c(s^{t+a})\right)$$

Subjective probabilities $\pi^{a,h}(s^{t+a}|s^t)$ may vary with age a and type h and may differ from "true" probability $\Pi(s^{t+a}|s^t)$

Financial Securities

At each date-event s^t, S Arrow securities in zero net supply, price vector $q(s^t) \in \mathbb{R}^S$

Lucas tree in unit net supply paying dividends $d(s^t) = d(s_t)$ traded at price $p(s^t)$

Aggregate endowment in the economy

$$\omega(s^t) = \omega(s_t) = d(s_t) + \sum_{a=1}^{N} \sum_{h=1}^{H} e^{a,h}(s_t)$$

Markets are dynamically complete

Financial Markets Equilibrium

Consumption $c^{a,h}(s^t)$

Portfolio of Arrow securities $\theta^{a,h}(s^t)$

Stock holdings $\phi^{a,h}(s^t)$

Equilibrium is a collection of prices and choices of individuals

$$\left(q(s^t), p(s^t), \left(\theta^{a,h}(s^t), \phi^{a,h}(s^t), c^{a,h}(s^t)\right)_{a=1,\ldots,N;h=1,\ldots,H}\right)_{s^t}$$

such that markets clear and agents optimize

Equilibrium: Existence and Uniqueness

Stock in positive net supply implies absence of bubbles
(Santos and Woodford, 1997)

Sequential equilibria are Arrow-Debreu equilibria

Arrow-Debreu equilibria exist
(Geanakoplos and Polemarchakis, 1991)

Value of aggregate endowment is finite; log utility implies gross substitute property, so A-D equilibrium is unique
(Kehoe et al., 1991)

Description of Equilibrium

Natural endogenous state variables: beginning-of-period cash-at-hand of agents of ages $a = 2, \ldots, N-1$

Beginning-of-period cash-at-hand $\kappa^{a,h}(s^t)$ at date-event s^t

$$\kappa^{a,h}(s^t) = \phi^{a-1,h}(s^{t-1})(p(s^t) + d(s^t)) + \theta_{s_t}^{a-1,h}(s^{t-1})$$

Equilibrium allocations and asset prices are linear functions of these state variables

Linear Recursive Equilibria

Consumption of the agent of age $a = 1, \ldots, N-1$, and type $h = 1, \ldots, H$, is a linear function of the individual cash-at-hand positions,

$$c^{a,h}(s^t) = \alpha_{1s}^{a,h} + \sum_{j=2}^{N-1} \sum_{i=1}^{H} \alpha_{jis}^{a,h} \kappa^{j,i}(s^t),$$

for some coefficients $\alpha_{jis}^{a,h} \geq 0$

The price of the tree is also a linear function of the individual cash-at-hand positions,

$$p(s^t) = \beta_{1s} + \sum_{a=2}^{N-1} \sum_{h=1}^{H} \beta_{ahs} \kappa^{a,h}(s^t),$$

for some coefficients $\beta_{ahs} \geq 0$

Linear Recursive Equilibria cont'd

The riskless rate R^f satisfies the relation

$$1/R^f(s^t) = \gamma_{1s} + \sum_{a=2}^{N-1} \sum_{h=1}^{H} \gamma_{ahs} \kappa^{a,h}(s^t),$$

for some coefficients $\gamma_{ahs} \geq 0$

Linear functions are not closed-form solutions

Tree price expression is a fixed-point equation

Expressions build foundation for a Numerical Procedure

Deterministic Dividends and Endowments

Dividends and endowments are independent of exogenous shock

$$e^{a,h}(s) = e^{a,h} \qquad d(s) = d$$

Only beliefs (may) depend on shock s

Now coefficients in linear consumption and pricing functions are independent of the shock s and thus the beliefs

Price of the Lucas tree

$$p(s^t) = \beta_1 + \sum_{a=2}^{N-1} \beta_a \sum_{h=1}^{H} \kappa^{a,h}(s^t)$$

for some coefficients β_a, $a = 1, \ldots, N-1$

OLG Economy with Analytical Solution

Follow specification of Huffman (1987)

$H = 1$ type per generation

No shocks, the tree is the only asset

No endowment after first period of life

$$e^{1,1} = e^1 = 1, \quad e^a = 0 \text{ for } a = 2, 3, \ldots, N$$

Closed-form solution for coefficients of tree price function

$$\beta_1 = \frac{\delta - \delta^N}{1 - \delta^N}, \quad \beta_a = \frac{\delta - \delta^{N-a+1}}{1 - \delta^{N-a+1}}, \text{ for } a = 2, \ldots, N-1,$$

for $\delta \neq 1$

Asset Prices in the Simple Economy

For $\delta = 1$ the tree price is

$$p(s^t) = \frac{\frac{N-1}{N} + d\left(\sum_{a=2}^{N-1} \frac{N-a}{N-a+1}\phi^{a-1}(s^t)\right)}{1 - \sum_{a=2}^{N-1} \frac{N-a}{N-a+1}\phi^{a-1}(s^t)}$$

If entire tree is held by agents of particular age a then

$$p(s^t) = (N-a)(1+d) + \frac{a-1}{N}.$$

If the entire tree is held by agents of age N then

$$p(s^t) = \beta_1 = \frac{N-1}{N}$$

Similar expressions for price of riskless bond

Asset Prices

Model with $N = 240$ periods, dividend $d = 1/2 = \frac{1}{3}\left(d + e^1\right)$

Prices $p(s^t)$ and $1/R^f(s^t)$ if agents of age a hold the entire Lucas tree

a	2	5	10	100
$p(s^t)$	357.00	352.52	345.04	210.41
$1/R^f(s^t)$	1.0028			

a	200	230	239	240
$p(s^t)$	60.829	15.954	2.4917	0.99583
$1/R^f(s^t)$	1.0028			0.0027855

Asset Price Volatility

Simple economy can be parameterized to obtain following result

Given any tree-price volatility, $\bar{v} < \infty$, and any bond-price volatility, $\underline{v} > 0$, for any time horizon $T > 1$ and any initial condition $\kappa \gg 0$, we can construct an economy where the stock price volatility is at least $\bar{v}$ while the bond-price volatility is at most $\underline{v}$, that is,

$$Std_{s,\kappa}^{T}(p) \geq \bar{v}, \quad Std_{s,\kappa}^{T}(1/R^{f}) \leq \underline{v} \ .$$

Idea: Make N sufficiently large and choose beliefs appropriately

Complete vs. Incomplete Markets

Price volatility can be arbitrarily large in an OLG model with a complete set of Arrow securities

OLG economy with many states, a single tree and no other securities has steady state (Huffman, 1987)

Consumption and savings decisions are independent of beliefs and only depend on discount factor and age of the agent

Without Arrow securities there is no complex trading and zero asset price volatility in the long run

Rich set of financial assets leads to a huge increase in the volatility of tree price

Effects of Intra-generational Belief Heterogeneity

Income profile in benchmark model clearly unrealistic

Question: How large is asset price volatility in an OLG economy with a properly estimated income process?

Model with 240 quarters (60 years)

Life-cycle income as estimated by Gourinchas and Parker (2002)

Data for Computational Experiment

As before, no uncertainty in endowments and dividends

Individual endowments $e^{a,h}$ sum to 2

Dividends $d = 1 = 0.15\left(d + \sum_{a,h} e^{a,h}\right)$

Discount factor $\delta \in \{0.99, 1.0, 1.01\}$

(Gourinchas and Parker, 2002, estimate 0.9924)

$S = 2$ i.i.d. and equi-probable shocks per period

$$\Pi(1,1) = \Pi(1,2) = \Pi(2,1) = \Pi(2,2) = \frac{1}{2}$$

Beliefs

$H = 3$ types of agents per generation

Fraction λ of agents has correct beliefs

Subjective beliefs of type 1 agents

$$\pi^{a,1} = \Pi$$

Subjective beliefs of type 2 agents

$$\pi^{a,2}(1,1) = \pi^{a,2}(2,1) = 1/2 + \epsilon,\ \pi^{a,2}(1,2) = \pi^{a,2}(2,2) = 1/2 - \epsilon$$

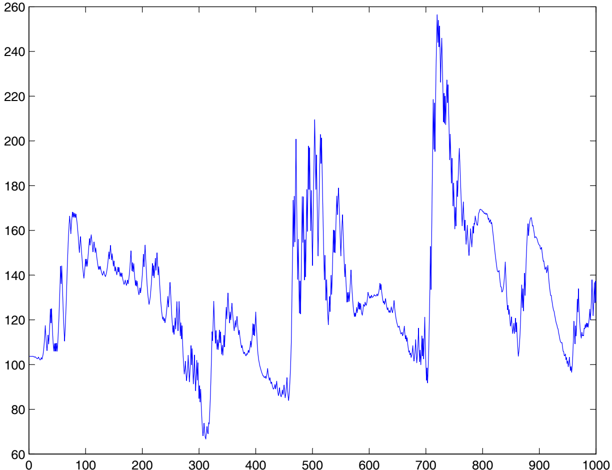
and of type 3 agents

$$\pi^{a,3}(1,1) = \pi^{a,3}(2,1) = 1/2 - \epsilon,\ \pi^{a,3}(1,2) = \pi^{a,3}(2,2) = 1/2 + \epsilon$$

Typical Simulation of OLG Economy over 1000 periods

Starting from the steady state with identical beliefs Π

Take $\lambda = 0,\ \varepsilon = 0.2,\ \delta = 1$ and get...



Long-run Volatility in Simulation

Volatility (in %) of bond and stock returns over 400,000 periods

δ	λ	$\varepsilon = 0.1$		$\varepsilon = 0.2$		$\varepsilon = 0.3$		$\varepsilon = 0.4$	
0.99	0	0.58	3.07	0.71	4.60	0.35	3.93	0.10	2.61
0.99	0.3	0.35	1.73	0.55	3.39	0.64	3.87	0.65	4.04
0.99	0.5	0.26	1.33	0.34	2.41	0.37	2.83	0.41	2.95
0.99	0.9	0.01	0.55	0.01	0.82	0.01	0.95	0.13	0.98
1	0	0.67	5.26	0.77	8.59	0.48	8.81	0.11	6.78
1	0.3	0.41	2.88	0.65	5.73	0.76	6.71	0.78	7.00
1	0.5	0.31	2.20	0.41	4.17	0.45	5.05	0.50	5.27
1	0.9	0.01	0.89	0.02	1.41	0.01	1.74	0.01	1.82
1.01	0	1.13	8.69	1.48	16.08	1.01	17.62	0.18	14.31
1.01	0.3	0.61	4.40	0.83	8.17	0.94	9.34	0.97	9.47
1.01	0.5	0.46	3.33	0.55	6.11	0.59	7.20	0.67	7.34
1.01	0.9	0.19	1.29	0.22	2.18	0.18	2.60	0.21	2.76

Long-run Volatility in Simulation

A few smart guys increase volatility...

δ	λ	$\varepsilon = 0.1$		$\varepsilon = 0.2$		$\varepsilon = 0.3$		$\varepsilon = 0.4$	
0.99	0	0.58	3.07	0.71	4.60	0.35	3.93	0.10	2.61
0.99	10^{-3}	0.57	3.04	2.47	8.84	3.03	9.90	3.53	8.98
1	0	0.67	5.26	0.77	8.59	0.48	8.81	0.11	6.78
1	10^{-3}	0.67	5.22	2.69	12.81	3.27	14.51	3.83	13.04
1.01	0	1.13	8.69	1.48	16.08	1.01	17.62	0.18	14.31
1.01	10^{-3}	1.29	8.37	3.14	17.26	3.50	18.46	3.81	16.50

Cash-at-Hand Shares

Aggregate all agents' shares of beginning-of-period cash at hand $\kappa^{a,h}(s^t)$ into 10 groups

Group 1,2, ..., 10 has respective cash-at-hand share

$$\frac{\sum_{h=1}^{3}\sum_{a=1}^{24}\kappa^{a,h}(s^t)}{p(s^t)+d(s^t)}, \frac{\sum_{h=1}^{3}\sum_{a=25}^{48}\kappa^{a,h}(s^t)}{p(s^t)+d(s^t)}, \dots, \frac{\sum_{h=1}^{3}\sum_{a=217}^{240}\kappa^{a,h}(s^t)}{p(s^t)+d(s^t)}$$

Benchmark economy $\lambda = 0.3,\ \varepsilon = 0.2,\ \delta = 1$

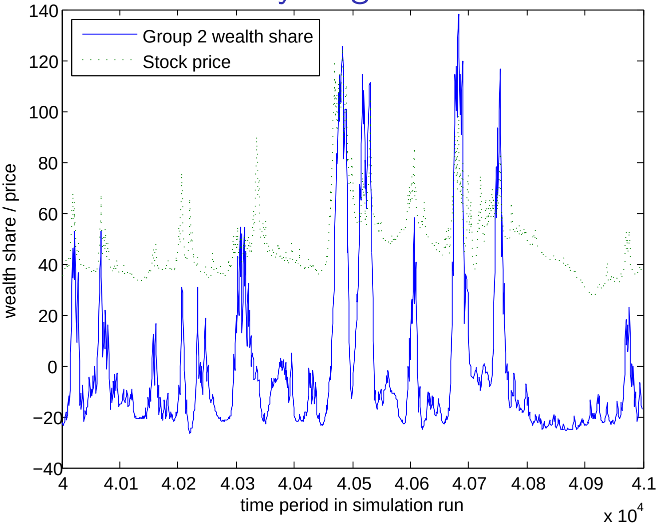
Cash-at-Hand Shares

Group	1	2	3	4	5
average (%)	1.17	-5.55	-8.47	-5.54	2.24
std. dev. (%)	18.29	26.21	22.10	17.26	15.49
$r(p)$	0.3579	0.6228	0.5554	0.3241	0.0458
Group	6	7	8	9	10
average (%)	12.99	24.99	33.01	30.30	14.94
std. dev. (%)	15.34	16.53	17.63	15.84	8.21
$r(p)$	-0.2519	-0.5050	-0.6633	-0.7341	-0.7203

Table: Wealth distribution – persistent differences in beliefs

When the young are rich, the stock price is high. When the old are rich, the stock price is low

When the young are rich....



Cash-at-Hand Shares for $\lambda = 0$

Group	1	2	3	4	5
average (%)	28.33	31.10	18.46	9.45	4.37
std. dev. (%)	46.97	56.55	51.29	45.18	38.54

Group	6	7	8	9	10
average (%)	2.43	2.38	2.27	0.97	0.24
std. dev. (%)	32.28	26.66	20.49	13.41	5.45

Table: Moments of wealth distribution, $\lambda = 0$ – persistent differences in beliefs

Converging beliefs

Suppose beliefs converge within generation:

$$\pi^{a,2}(1,1) = \pi^{a,2}(2,1) = \left(\frac{1}{2} + \varepsilon\right)\left(1 - \frac{a}{240}\right) + \frac{1}{2}\frac{a}{240},$$

and

$$\pi^{a,3}(1,1) = \pi^{a,3}(2,1) = \left(\frac{1}{2} - \varepsilon\right)\left(1 - \frac{a}{240}\right) + \frac{1}{2}\frac{a}{240},$$

Take $\delta = 1$ and $\lambda = 0.3$.

Case	$\varepsilon = 0.1$		$\varepsilon = 0.2$		$\varepsilon = 0.3$		$\varepsilon = 0.4$	
Conv.	0.45	2.28	0.74	5.16	0.70	6.51	0.78	6.89
Stat	0.41	2.88	0.65	5.73	0.76	6.71	0.78	7.00

Table: Volatility (Std(R^f) and Std(R^e) in %) – converging beliefs

Temporary Disagreement

Fraction λ of agents always holds correct beliefs (type 1)

Type 2 and 3 have correct beliefs most of the time, but regime switches lead to temporary disagreement

$S = 3$ shocks, true law of motion

$$\Pi = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.4 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

In shocks 2 and 3, for $a = 1, \dots, N-1$, $h = 2, 3$,

$$\pi^{a,h}(2,1) = \pi^{a,h}(3,1) = 0.2,$$

$$\pi^{a,2}(2,2) = \pi^{a,2}(3,2) = 0.4 + \varepsilon$$

$$\pi^{a,3}(2,2) = \pi^{a,3}(3,2) = 0.4 - \varepsilon$$

Temporary Disagreement

Fraction $\lambda = 0.3$ of type 1 agents

Fraction $(1 - \lambda)/2 = 0.35$ of agents are of type 2 and 3, resp.

Volatility (in %) of bond and stock returns over 400,000 periods

$\lambda \backslash \epsilon$	0.1		0.2		0.3	
0.3	0.46	2.35	0.70	4.41	0.78	5.21

Temporary disagreement suffices for reasonably large volatility

Aggregate Uncertainty and Identical Beliefs

Until now: No uncertainty in endowments and dividends

With identical beliefs: Steady state with zero price volatility

Now: Aggregate uncertainty but identical beliefs

Rios-Rull (1996), Storesletten et al. (2007):
Wealth distribution changes very little in OLG economies with aggregate uncertainty and identical beliefs

In our model there exist sufficient conditions for a stochastic steady state

Stochastic Steady State

Consider an economy where all agents $a = 1, \dots, N$, $h = 1, \dots, H$, have identical and correct beliefs, $\pi^{a,h} = \Pi$. Then, under either of the following two assumptions, there exist initial conditions κ such that in the resulting equilibrium, prices and consumption choices are time invariant functions of the exogenous shock alone.

1. All endowments and dividends are collinear, i.e. for all agents $a = 1, \dots, N$, $h = 1, \dots, H$, it holds that

$$\frac{e^{a,h}(s)}{e^{a,h}(s')} = \frac{d(s)}{d(s')} \quad \text{for all} \quad s, s' = 1, \dots, S.$$

2. Shocks are i.i.d., i.e. for all shocks s', $\Pi(s, s')$ is independent of s, and endowments of all agents of age $a = 1$ are collinear to aggregate endowments, i.e. for all $h = 1, \dots, H$,

$$\frac{e^{1,h}(s)}{e^{1,h}(s')} = \frac{\omega(s)}{\omega(s')} \quad \text{for all} \quad s, s'.$$

Properties of Typical Calibrations

Common calibrations do not satisfy assumptions of the stochastic steady state

- labor endowments are assumed to be safe, or
- shocks to labor endowments and dividends are independent

Does such a calibration of our OLG model lead to substantial price volatility?

Computational Experiment

$H = 1$ agent per generation

Large shocks for dividends and endowments

$d(1) = d(2) = 0.9,\ d(3) = d(4) = 1.1$

$e^a(1) = e^a(3) = 0.9e^a,\ e^a(2) = e^a(4) = 1.1e^a$
with labor endowments e^a, $a = 1, \ldots, 240$, as before.

$\Pi(s, s') = 1/4$ for all s, s'.

Large Exogenous Shocks – Small Price Volatility

s	1	2	3	4
Avg. price	93.71	100.66	107.59	114.54
Std. dev.	0.0018	0.0020	0.0021	0.0023

Model with homogeneous beliefs and aggregate uncertainty yields only tiny volatility (in line with previous literature)

Summary

Canonical and parsimonious stochastic OLG model with dynamically complete markets and heterogeneous beliefs

Assumption of log utility leads to linear recursive equilibria

Analytical example shows that asset prices depend on the wealth distribution

Calibrated OLG economy exhibits substantial tree price volatility

Summary cont'd

Mechanism:

small differences in belief

$\Longrightarrow$

large movements in the wealth distribution

$\Longrightarrow$

substantial asset price volatility

Arrow-Debreu model delivering substantial volatility

Model with homogeneous beliefs and aggregate uncertainty yields only tiny volatility (in line with previous literature)

Sufficient conditions for stationary wealth distribution and prices to depend only on exogenous shock

Many Open Questions

Belief heterogeneity has strong impact on intra- and inter-generational wealth distribution

Changes in the wealth distribution strongly affect asset prices

Question: Is there empirical evidence for these effects?

Market completeness enables agents to make "large bets"

Question: Welfare effects of complete vs. incomplete markets

Equilibrium Properties

Unique equilibrium consumption allocation

Equilibrium portfolios at each date-event s^t are a one-dimensional subspace of $\mathbb{R}^S$

One asset "too many": Lucas-tree and S Arrow securities

Numerical solution procedure exploits this multiplicity by imposing an additional restriction

Additional Restriction

Agent of age $N-1$ and type 1 must buy the entire Lucas-tree

She holds it for one period and sells it in her last, Nth, period

All other agents are only permitted to trade Arrow securities

Cash-at-hand position at date-event $s^{t+1} = (s^t, s_{t+1})$ for these agents

$$\kappa^{a+1,h}(s^{t+1}) = \theta^{a,h}_{s_{t+1}}(s^t)$$

Agent's cash-at-hand entering state s_{t+1} is just his holding of the Arrow security paying in that state

Equilibrium Equations

Generic first-order conditions for agents' utility maximization problems with respect to holdings of Arrow security for shock s'

$$-q_{s'}(s^t)u'(c^{a,h}(s^t)) + \delta\pi^{a,h}(s'|s)u'(c^{a+1,h}(s^t,s')) = 0$$

Substitution of linear policy functions yields for $a = 2, \ldots, N-2$:

$$\begin{aligned} 0 \quad = \quad & -q_{s'}\left(\alpha_{1s'}^{a+1,h} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\alpha_{jis'}^{a+1,h}\theta_{s'}^{j-1,i}\right) \\ & +\delta\pi^{a,h}(s'|s)\left(e_s^{a,h} + \kappa^{a,h} - \sum_{s'} q_{s'}\theta_{s'}^{a,h}\right). \end{aligned}$$

Similar expressions for other agents

Equilibrium Equations cont'd

Condition for tree price from Euler equation for agent of age $N-1$, type 1

Market clearing equations

Generic consistency equation for linear consumption functions

$$\alpha_{1s}^{a,h} + \sum_{j=2}^{N-1} \sum_{i=1}^{H} \alpha_{jis}^{a,h} \kappa^{j,i} = e_s^{a,h} + \kappa^{a,h} - \sum_{s'} q_{s'} \theta_{s'}^{a,h}$$

Grid of Initial Conditions

To determine coefficients of linear policy and pricing functions we need to vary initial conditions

Convenient choice for initial values $\kappa^{a,h}$ are the zero vector and all possible unit vectors

Large system of nonlinear equations (which cannot be solved)

Iterative Jacobi Method

For fixed coefficients α and β the system is linear in the prices of Arrow securities $q_{s'}$ and investments $q_{s'}\theta_{s'}^{a,h}$

Solve a linear (sub)system at all grid points for $q_{s'}$ and $q_{s'}\theta_{s'}^{a,h}$

Given the just computed solutions for $q_{s'}$ and $q_{s'}\theta_{s'}^{a,h}$ the new iterates for the coefficient vectors α and β are the solution of another linear (sub)system

Iterative procedure terminates when infinity norm of difference between last two iterates falls below 10^{-10}

Return to OLG Model