The History-Representation for Solving and Deriving Optimal Policies in Heterogeneous agent models with Aggregate Shocks

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Abstract

We present solution techniques to solve incomplete market models with aggregate shocks using a truncated representation for idiosyncratic histories. In this paper, we present a simple theory of a projection on the space of idiosyncratic histories, to present a finite-dimensional state space heterogeneous agent models. This approach allows us to improve current algorithms for solving such models with aggregate shocks, as Reiter (2009) for instance. In addition, it provides tools to derive optimal Ramsey policies both at the steady state and with aggregate shocks. We compare the outcomes of these solution to those of alternative methods.

Keywords: Incomplete markets, numerical methods, Ramsey model, finite-state representation.

JEL codes: E21, E44, D91, D31.

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1 Introduction

Heterogeneous agent models, or more precisely incomplete-insurance market models, are now used for in many fields of economics, to study households inequality or firm heterogeneity for instance. Various algorithms are now used to solve these models with aggregate shocks, what is still a challenge. In addition, tools are missing to solve for optimal policies without and with aggregate shocks. Such tools would allow to understand distortions in these economies and provide normative insight about business cycle management.

In this paper, we present a simple theory of a projection on the space of idiosyncratic histories. It allows to improve on current algorithm to solve these models with aggregate shocks, such as Reiter (2009). In addition, this theory provides tools to derive optimal policies both at the steady state and with aggregate shocks.

The basic idea is the following, in heterogeneous agent models, agents are different because they have different histories of uninsurable idiosyncratic shocks. One can use a time-invariant partition of these histories $\mathcal{P}$ such that each agent, at each period, belongs to one and exactly one element of this partition. The key aspect is the proper choice of the partition. An explicit partition is used in LeGrand and Ragot (2017), based on a truncation of idiosyncratic histories for the last N periods. Each agent having the history of the idiosyncratic shock for the last N period are in the same element of $\mathcal{P}$. Implicit partitions can be provided using Reiter (2009) methodology, which is to use the steady-state distribution of wealth to identify agents: Defining a partition in the support of the distribution of wealth one implicitly define elements of the partition $\mathcal{P}$ as the histories generating amount of in the same partition of wealth. The Bewley model can then be approximated following the finite elements of the partition $\mathcal{P}$ instead of infinite support of the distribution of the Bewley model.

The interest of this construction is threefold. First, constructing the projection, we show that it improves on current algorithm using projection methods to solve model with aggregate shocks. It indeed allows to use extract more information about the steady-state distribution of the Bewley model to simulate the economy with aggregate shocks. Heterogeneity in Euler equations with each elements of the partition $\mathcal{P}$ is captured by a new parameter, improving the algorithm. For instance, we show that a standard economy with uninsurable employment risk and aggregate technology shock is accurately simulated with only 22 agents. In addition, this construction provides a simple algorithm to solve models with time-varying idiosyncratic risk, which is a difficult task with other algorithms.

Second and more importantly, this construction allows deriving optimal Ramsey program with aggregate shocks. The basic idea is the following, one can use tools developed in dynamic contract, namely Marcet and Marimon (2011) applied on elements of the partition $\mathcal{P}$, to derive first-order conditions for the planner. These conditions are then easy to simulate with aggregate shocks. In these economies, the difficult part is to find the steady state of the optimal Ramsey policies (and check that this interior solution is consistent with second order conditions). The projection techniques provides a simple algorithm using information from a general Bewley model, to show the convergence of the instrument of the planner. We apply this methodology to a simple problem, the time-varying provision of public good in a economy with uninsurable employment risk and aggregate productivity shock, where the public good is financed by a non-distorting tax on labor. This example is, on purpose, the simplest one to present the methodology and to discuss the difference between complete and incomplete markets.

A third interest of this construction is to provide a theoretical representation of algorithm using projections methods. The gain is that equations, such as the first-order conditions of the planner, are easy to understand economically. This helps to identify the effects in these very complex models.

This paper is mainly related to two strands of the literature. The first one is one the computation of incomplete insurance markets with aggregate shocks. The second one, much smaller, is on optimal Ramsey policies in these models. After the seminal paper of Krusell and Smith (1998), incomplete insurance market models with aggregate shocks have first been solved using a fixed point on simple rules to form expectations, moments to introduce at elements used to approximate rational expectations have generated a literature on approximate aggregation. After the work of Reiter (2009) and Algan, Allais, and Den Hann (2010) the literature has moved toward projection methods to first simplify the distribution of wealth and then simulate the model. These techniques are now used in various setups, to solve discrete time models Winberry (2016) or models first written in continuous time or Ahn, Kaplan, Moll, Winberry, and Wolf (2017). In this literature, our contribution is to improve on simple projections methods by using more information about the steady-state Bewley model, which is the heterogeneity in the Euler equation among elements n which the Bewley model is projected.

Second, this paper is related to the literature on optimal (Ramsey) policies in heterogeneous agent models. This literature is very thin and very recent. First, Açikgöz (2015) provides an algorithm to solve for the steady-state allocation of the Ramsey program, based on assumptions on functional form. Nuño and Moll (2017) use a continuous-time approach without aggregate chocks and then use projections methods to find the steady state allocation. Bhandari, Evans, Golosov, and Sargent (2016) present a solution method of models with aggregate shocks, which relies on perturbation methods around time-varying allocations. They solve the model approximating the distribution with a bery large number of agents. LeGrand and Ragot (2017) derive optimal Ramsey policy using a truncation in the space of idiosyncratic histories. Compared to these models our contribution is to provide a general representation allowing to simulate models with optimal policies and aggregate shock, which is much more general than truncation approach

Section 2 presents the simple environment, on which our methodology will be applied. Section 3 presents the projection in the space of idiosyncratic histories in the general case. Section 4 presents solution techniques to derive optimal policies. Section 5 analyses in more detail how Reiter (2009) can be understood as an implicit partition to provide some improvement on its algorithm. Section 6 provides two numerical examples, a first one without optimal policies to benchmark our method with other ones presented in the literature. The second one computes optimal time-varying fiscal policy.

2 The economy

We consider a discrete-time setup. The economy features a single good and is populated by a population of size 1 of agents distributed on a segment I according to a measure $l(\cdot)$.

2.1 Preferences

Agents derive utility in each period from private consumption c and from the provision of a public good G. The period utility function is denoted $U(c,G)$. We assume that the period utility function is separable in private consumption and public good provision. More precisely, the function $U(c,G)$ is supposed to have the following functional form:

$$U(c,G) = u(c) + v(G),$$

where u and v are twice continuously derivable functions from $\mathbb{R}_+$ onto $\mathbb{R}$. Functions u and v are assumed to be strictly increasing and concave, with $\lim_{c\to 0+} u'(c) = \infty$.

In what follows, we use a CRRA utility function:

$$u(c) = \frac{c^{1-\gamma}-1}{1-\gamma} + \chi\frac{G^{1-\gamma_G}-1}{1-\gamma_G}, \tag{1}$$

where $0 < \gamma, \gamma_G \neq 1$. When $\gamma = \gamma_G = 1$, the utility function is simply $U(c,G) = \log(c) +$

$\chi \log(G)$ (the two other cases $\gamma = 1 \neq \gamma_G$ and $\gamma \neq 1 = \gamma_G$ are straightforward to deduce).

Agents have standard additive intertemporal preferences, with a constant discount factor $\beta > 0$. They therefore rank consumption and public good streams, denoted respectively by $(c_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$, using the intertemporal criterion $\sum_{t=0}^{\infty} \beta^t U(c_t, G_t)$.

2.2 Risks

Aggregate risk. The aggregate risk is represented by a probability space $(\mathcal{S}^\infty, \mathcal{F}, \mathbb{P})$. At a given date t, the aggregate state is denoted s_t and takes values in the state space $\mathcal{S} \subset \mathbb{R}^+$. We assume the aggregate risk to be a first-order Markov process. The history of aggregate shocks up to time t is denoted $s^t = \{s_0, \ldots, s_t\} \in \mathcal{S}^{t+1}$. Finally, the period-0 probability density function of any history s^t is denoted $m^t(s^t)$.

For the sake of clarity, for any random variable $X_t : \mathcal{S}^t \to \mathbb{R}$, we will denote X_t, instead of $X_t(s^t)$, its realization in state s^t,

Employment risk. At the beginning of each period, agents face an uninsurable idiosyncratic employment risk, denoted e_t at date t. The employment status e_t can take two values, 0 and 1, corresponding to employment and unemployment respectively. Employed agents with $e_t = 1$ can supply inelastically one unit of labor, and they earn a before-tax real wage, denoted w_t at date t. Unemployed agents with $e_t = 0$ cannot work and will receive unemployment benefits, denoted ϕw_t at date t. The quantity $\phi > 0$ measures the replacement rate.

The employment status e_t follows a discrete first-order Markov process with transition matrix $M_t \in [0,1]^{2\times 2}$. The job separation rate between periods $t-1$ and t is denoted s_t, while f_t is the job finding rate between $t-1$ and t. The time-varying transition matrix across employment status is therefore:

$$M_t = \begin{bmatrix} 1 - f_t & f_t \\ s_t & 1 - s_t \end{bmatrix}. \tag{2}$$

The history of idiosyncratic shocks up to date t is denoted $e^t = \{e_0, \ldots, e_t\} \in \{0,1\}^{t+1} = \mathcal{E}^{t+1}$.

2.3 Production

The good is produced by a unique profit-maximizing representative firm. This firm is endowed with a production technology that transforms, at date t, labor L_t and capital K_{t-1} into Y_t output units of the single good. The production function is a Cobb-Douglas function with parameter $\alpha \in (0,1)$ featuring constant returns-to-scale. The capital must be installed one period before production and the total productivity factor A_t is stochastic. Denoting as $\delta > 0$ the constant capital depreciation, the output Y_t is formally defined as follows:

$$Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1}. \tag{3}$$

The logarithm of the total productivity factor is assumed to be an AR(1) process with autocorrelation ρ_A and variance of the innovation σ_A^2, such that:

$$\begin{aligned} A_t &= \exp(a_t), \qquad (4) \\ \text{with: } a_t &= \rho_A a_{t-1} + \varepsilon_t^A \text{ and } \left(\varepsilon_t^A\right)_t \sim_{\text{iid}} \mathcal{N}\left(0, \sigma_A^2\right). \end{aligned}$$

The two factor prices at date t are the before-tax wage rate $\tilde{w}_t$ and the capital return r_t. As we explain further below, we assume that while labor is taxed at a linear rate, capital is not taxed. The profit maximization of the producing firm implies the following factor prices.

$$\tilde{w}_t = (1 - \alpha) A_t \left(\frac{K_{t-1}}{L_t}\right)^{\alpha}, \tag{5}$$

$$r_t = \alpha A_t \left(\frac{K_{t-1}}{L_t}\right)^{\alpha-1} - \delta. \tag{6}$$

2.4 Social contributions and taxes

The government raises both social contributions and capital taxes, which have two distinct objectives. Social contributions solely serve to finance unemployment benefits, while capital tax serves to finance the public good.

Unemployment benefits. Social contributions amount to a constant proportion τ_t of the wage of employed agents. The contribution τ_t is set such that the unemployment insurance (UI) scheme is balanced at any date t. There is no possible social debt. The formal expression of the contribution τ_t depends on the population of employed agents. Since only employed agents supply labor and since their productivity is equal to one, the population of employed agents amounts to the aggregate labor supply L_t. Conversely, the total population size being normalized to one, the population of unemployed agents amounts to $1 - L_t$. Recalling that the replacement rate is ϕ, the balance of the UI scheme therefore implies:

$$\tau_t = \frac{1 - L_t}{L_t} \phi. \tag{7}$$

Fiscal policy. The labor tax τ_t^L finances a quantity G_t of public goods. The government is prevented from raising public debt, such that the government budget is balanced at any date. Formally, the government budget constraint can be expressed as:

$$G_t = \tau_t^L \tilde{w}_t L_t. \tag{8}$$

We follow Chamley (1986) and use the CRS property of the production function to express the budget constraint of the government in post-tax terms. The post-tax real wage rate, denoted w_t, verifies:

$$w_t = \left(1 - \tau_t^L\right) \tilde{w}_t. \tag{9}$$

Since the production function implies $Y_t = K_{t-1} r_t + L_t \tilde{w}_t$, the budget constraint of the government (8) can be rewritten as follow:

$$G_t + K_{t-1} r_t + L_t w_t \quad = \quad A_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1}. \tag{10}$$

Justification of the fiscal system. We have chosen the above setup, since it features one of the simplest fiscal system we can think of. It will simplify the comparison between complete and incomplete-market economies. Indeed, the labor tax is non-distorting as the labor supply is inelastic. As a consequence, the complete market allocation reproduces the first-best allocation. The difference between complete and incomplete market economies will only result from the distributional effect of labor tax.

2.5 Agents' program and resource constraints

2.5.1 Sequential formulation

We consider an agent i. She can save in a riskless asset that pays off the post-tax gross interest rate $1 + r_t$. She is prevented from holding too negative savings and the latter must remain greater than $-\bar{a}$. At date 0, the agent chooses her consumption $(c_t^i)_{t \geq 0}$ and her saving plans $(a_t^i)_{t \geq 0}$ that maximize her intertemporal utility, subject to a budget constraint and the previous borrowing limit. Formally, her program can be expressed as follows:

$$\max_{\{c_t^i, a_t^i\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t^i, G_t) \tag{11}$$

$$c_t^i + a_t^i = (1 + r_t) a_{t-1}^i + \left((1 - \tau_t) e_t^i + \phi(1 - e_t^i)\right) w_t, \tag{12}$$

$$a_t^i \geq -\bar{a}. \tag{13}$$

The budget constraint (12) is very standard and the expression $\left((1-\tau_t)e_t^i + \phi(1-e_t^i)\right) w_t$ is a compact formulation for the net wage of the agent, depending on whether she is employed ($e_t^i = 1$) or unemployed ($e_t^i = 0$). We now turn to the economy-wide constraints. First, the financial market clearing implies the following relationship:

$$\int_i a_t^i l(di) = K_t. \tag{14}$$

The clearing of goods market implies that the total consumption, made of private individual consumption, private firm consumption and public consumption equals total supply, made of output and past capital:

$$\int_i c_t^i l(di) + G_t + K_t = Y_t + K_{t-1}. \tag{15}$$

Finally, using the transition matrix M_t in equation (2), we deduce the law of motion for the labor supply:

$$L_t = (1 - s_t)\, L_{t-1} + f_t\, (1 - L_{t-1})\,. \tag{16}$$

We can finally formulate our equilibrium definition.

Definition 1 (Sequential equilibrium) *A sequential competitive equilibrium is a collection of individual allocations $\left(c_t^i, a_t^i\right)_{t\geq 0, i\in I}$, of aggregate quantities $(G_t, L_t, K_t)_{t\geq 0}$, of price processes $(\tilde{w}_t, w_t, r_t)_{t\geq 0}$, and of social contributions and capital taxes $(\tau_t, \tau_t^L)_{t\geq 0}$, such that, for an initial wealth distribution $\left(a_{-1}^i\right)_{i\in I}$, and for initial values of capital stock $K_{-1} = \int_{i\in I} a_{-1}^i l(di)$, of capital tax τ_0, and of the initial aggregate shock s_{-1}, we have:*

1. *given prices, individual strategies $\left(c_t^i, a_t^i\right)_{t\geq 0, i\in I}$ solve the agents' optimization program in equations (11)–(13);*

2. *financial and good markets clear at all dates: for any $t \geq 0$, equations (14) and (15) hold;*

3. *the government budget constraint (10) and the UI scheme balance (7) hold at any date;*

4. *factor prices $(\tilde{w}_t, w_t, r_t)_{t\geq 0}$ are consistent with (5), (6), and (9).*

2.5.2 Recursive formulation: Bewley model

We now express the previous program (11)–(13) using an equivalent recursive formulation in absence of aggregate shocks. From Huggett (1993), we know that such a recursive formulation exists in such a case. We use standard notations of dynamic programming. The value function is denoted V and it is known to depend on the beginning-of-period wealth, a and the individual state $e \in \{0,1\}$. With this notation, the recursive formulation of the program can be recursively written as follows:

$$V(a,e) = \max_{c,a \in \mathbb{R}} \; u(c) + v(G) + \beta \mathbb{E}\left[V(a',e')\right], \tag{17}$$

$$c + a' = w(e) + (1+r)a, \tag{18}$$

$$c \geq 0,\, a' \geq -\bar{a}, \tag{19}$$

where the prime characterizes next-period values.

A recursive equilibrium in this economy is: (i) a collection of policy rules, $g_c(a,e)$ for consumption, $g_a(a,e)$ for savings, $\nu(a,e)$ for the Lagrange multiplier on the credit limit, (ii) a distribution function for wealth levels $\Gamma(a,e)$, (iii) price processes w, $\tilde{r}$, and r, (iv) tax processes τ and τ^K, (v) aggregate quantities K, L, and Y such that: (i) the policy rules solve the agent problem (17)–(19); (ii) financial and good markets clear: $K' = \int_{a,e} g_a(a,e)\Gamma(da,de)$, $L = \int_{a,z} e\Gamma(da,de)$, $Y = F(K,L)$, and $\int_{a,e} g_a(a,e)\Gamma(da,de) + G + K' = Y + K$; (iii) prices are set competitively: $\tilde{r} = F_K(K,L)$, $r = (1-\tau^K)r$ and $w = F_L(K,L)$; (iv) government and UI budget constraints hold: $\tau = \frac{1-L}{L}\mu$ and $G + K'r + Lw = Y - \delta K$.

3 Solving the model with history representation

The previous equilibrium representation features idiosyncratic histories of infinite length. We now show how we represent the previous equilibrium using finite length histories. Loosely speaking, we will project the previous policy on a finite state space.

3.1 Partitions

A finite history of length $N \geq 1$ is a vector $h = (h_{-N+1}, \ldots, h_0) \in \{0,1\}^N$ of length N representing the realizations of idiosyncratic shocks over the N consecutive previous periods. For instance, h_0 is the current idiosyncratic realization, h_1 is the realization one period ago, and h_{-N+1} the realization $N-1$ periods ago. A finite collection of histories of different length, denoted $\mathcal{P}$, will be called a partition if every element of $\{0,1\}^\infty$ can be associated to a unique history of the partition. In other words, the history of every agent in the economy is projected onto one, and exactly one, history in the set $\mathcal{P}$. The cardinal of $\mathcal{P}$ is denoted P. The most simple example of partition is $(\{e\},\{u\})$ and corresponds to a case, where the history of every agent is summarized by her current idiosyncratic status. This partition will be said to be the *minimal* partition.

Even though it has not been formalized in those terms, partitions have already been used in the literature. First, Challe, Matheron, Ragot, and Rubio-Ramirez (2017) use a three-state partition, which is $(\{e\},\{eu\},\{uu\})$. In each period, any agent can be in one and only one of these three states: employed, unemployed now and in the previous period, or unemployed now and employed before. This three-state partition is shown to be sufficient to capture time-varying precautionary savings. The transition matrix between these three states is easy to derive from labor market transitions. For instance $\Pi_{t,\{e\},\{eu\}} = s_t$, $\Pi_{t,\{eu\},\{e\}} = \Pi_{t,\{uu\},\{e\}} = f_t$, and finally $\Pi_{t,\{eu\},\{uu\}} = 1 - f_t$, where we recall that s_t and f_t are the job-transition probabilities –see equation (2). Second, LeGrand and Ragot (2017) use a more general truncation space. For a given parameter N, the partition, denoted $\mathcal{P}^N$, contains all idiosyncratic histories of length N, or more formally all vectors $(e_{N-1}, \ldots, e_0) \in \{0,1\}^N$. In this case, the transition matrix between partition elements can be easily derived from the transition matrix M_t. Note that even though the transition matrix can be time-varying, the partition remains constant over time.

An history $h = (e_{N-1}, \ldots, e_0)$ will be said to be a *truncation* of an history $h' = (e'_{N'-1}, \ldots, e'_0)$ and will be denoted $h' \succeq h$ if $N' \geq N$ and $e_i = e'_i$ for all $i = 0, \ldots, N-1$. Conversely, the history h' will be said to be a *prolongation* of h. It is noteworthy, the binary relation $\succeq$ defines a preorder on the set of histories. In particular, $\succeq$ is transitive. In words, if $h' \succeq h$, the history h drops the oldest elements of h' and only keeps the N last ones. A partition $\mathcal{P}^A$ will said to be finer than a partition $\mathcal{P}^B$ if every element of $\mathcal{P}^A$ is a prolongation of an element of $\mathcal{P}^B$. We will note it $\mathcal{P}^B \subset \mathcal{P}^A$. Conversely, we will say that $\mathcal{P}^B$ is coarser than $\mathcal{P}^A$. For instance, the three-state partition $(\{e\},\{eu\},\{uu\})$ that we have seen above is finer than the minimal partition $(\{e\},\{u\})$. In the remainder of the paper, we will only consider partitions, that are finer than the minimal partition.

3.2 From the Bewley model to an history representation

Projecting the Bewley model on a partition $\mathcal{P}$. Let consider a partition $\mathcal{P}$ of histories as discussed above. We will construct a finite history representation of the Bewley model on $\mathcal{P}$ as follows –we will say that we project the Bewley model on $\mathcal{P}$. The idea will consist in "pooling" the choices of agents with the same history h. We start with the probability distribution of agents with history h and asset holding a, denoted $\Gamma^{\mathcal{P}}(a,h)$ defined over $A \times \bigcup_{t=1}^{\infty}\{0,1\}^t$, where $A = [-\overline{a}, \infty)$ is the saving space. This probability can be defined by iteration of the probability distribution Γ for the Bewley model. Second, we can deduce the size of the population of agents with history h, that we denote S_h:

$$S_h = \int_A \Gamma^{\mathcal{P}}(da, h), \tag{20}$$

which is simply the measure of agents with history h, independently of their asset holdings.

We now turn to the policy functions and asset choices. The beginning-of-period asset holding, denoted $\tilde{a}_h$ is:

$$\tilde{a}_h = \int_A a \frac{\Gamma^{\mathcal{P}}(da, h)}{S_h},$$

which is the average asset holding among the population with history h. The asset choice, a'_h, which is the average end-of-period asset holding of agents with history h can be expressed as:

$$a'_h = \int_A g_a(a, e_0(h)) \frac{\Gamma^{\mathcal{P}}(da, h)}{S_h}, \tag{21}$$

where $e_0 : h \in \mathcal{P} \to e_0(h) \in \{0, 1\}$ returns the current idiosyncratic state for any history h. We proceed similarly for the average consumption choice –denoted c_h– and the average Lagrange multiplier of the credit constraint –denoted ν_h. We obtain the following expressions:

$$c_h = \int_A g_c(a, e_0(h)) \frac{d\Gamma^{\mathcal{P}}(da, h)}{S_h}, \tag{22}$$

$$\nu_h = \int_A \nu(a, e_0(h)) \frac{d\Gamma^{\mathcal{P}}(da, h)}{S_h}. \tag{23}$$

Note that ν_h is positive if and only if a positive measure of agents having history h face binding credit constraints.

Projection on a coarser partition. We will now define a projection from an initial partition $\mathcal{P}^1$ to a coarser partition $\mathcal{P}^2$, with $\mathcal{P}^1 \subset \mathcal{P}^2$. We will check that projecting on a partition $\mathcal{P}^2$ from a finer partition $\mathcal{P}^1$ is similar to projecting directly the Bewley model on $\mathcal{P}^2$. In this section, we will use the superscripts $\mathcal{P}^1$ and $\mathcal{P}^2$ for characterizing variables in $\mathcal{P}^1$ and $\mathcal{P}^2$, respectively. First, the probability distribution $\Gamma^{\mathcal{P}^2}$ can be deduced from $\Gamma^{\mathcal{P}^1}$. Indeed, by construction of partitions, any history $h_2 \in \mathcal{P}^2$ that is a truncation of histories $h_1 \in \mathcal{P}^1$ ($h_1 \succeq h_2$), we have:

$$\Gamma^{\mathcal{P}^2}(a, h_2) = \sum_{h_1 \succeq h_2} \Gamma^{\mathcal{P}^1}(a, h_1).$$

The size of the population of agents with history $h_2 \in \mathcal{P}^2$, denoted $S_{h_2}^{\mathcal{P}^2}$ is:

$$S_{h_2}^{\mathcal{P}^2} = \sum_{h_1 \succeq h_2} S_{h_1}^{\mathcal{P}^1}.$$

We can check from equation (20) that $\sum_{h_1 \succeq h_2} S_{h_1}^{\mathcal{P}^1} = \int_A \sum_{h_1 \succeq h_2} d\Gamma^{\mathcal{P}^1}(da, h_1) = \int_A d\Gamma^{\mathcal{P}^2}(a, h_2)$. This makes it clear that projecting from $\mathcal{P}^1$ onto $\mathcal{P}^2$ is similarly to directly projecting the Bewley model onto $\mathcal{P}^2$. For the consumption policy function, we have:

$$c_{h_2}^{\mathcal{P}^2} = \sum_{h_1 \succeq h_2} c_{h_1}^{\mathcal{P}^1} \frac{S_{h_1}^{\mathcal{P}^1}}{S_{h_2}^{\mathcal{P}^2}}.$$

We deduce from the expression of c_h in (22) that:

$$\begin{aligned} c_{h_2}^{\mathcal{P}^2} &= \sum_{h_1 \succeq h_2} \int_A g_c(a, e_0(h_1)) \frac{\Gamma^{\mathcal{P}^1}(da, h_1)}{S_{h_1}^{\mathcal{P}^1}} \frac{S_{h_1}^{\mathcal{P}^1}}{S_{h_2}^{\mathcal{P}^2}}, \\ &= \int_A g_c(a, e_0(h_2)) \sum_{h_1 \succeq h_2} \Gamma^{\mathcal{P}^1}(da, h_1) \frac{1}{S_{h_2}^{\mathcal{P}^2}}, \\ &= \int_A g_c(a, e_0(h_2)) \frac{\Gamma^{\mathcal{P}^2}(da, h_2)}{S_{h_2}^{\mathcal{P}^2}}, \end{aligned}$$

where the second equality comes from the fact that h_2 and h_1 share the same current state: $e_0(h_1) = e_0(h_2)$. We can proceed similarly for the saving choice $a_{h_2}^{\mathcal{P}^2}$ and the Lagrange multiplier $\nu_{h_2}^{\mathcal{P}^2}$.

Projecting on a coarser partition will enable us to to consider different history sizes. In quantitative applications – see Section 6 –, it turns out that a fine partition is useful for computing a precise steady-state, while a coarser partition enables to accurately depict the dynamics. The projection of the fine partition onto the coarse one enables to have two representations that are consistent with each other.

Credit-constrained agents. We assume that the partition $\mathcal{P}$ contains a set of histories $\mathcal{P}^{cc}$, such all credit constrained agents –and only them– have an history that belongs to $\mathcal{P}^{cc}$. The existence of this set derives from the assumption that credit constraints are above the natural borrowing limits. We show below in Section 5 that this set can be easily derived quantitatively from the steady-state outcomes of the Bewley model.

Assumption A (Credit constrained histories) *There exists a subset of histories $\mathcal{P}^{cc} \subset \mathcal{P}$ which only gathers the histories of every credit constrained agent.*

Every history in $\mathcal{P}^{cc}$is therefore an history that leads an agent to a binding credit constraint. As a direct implication, we have $\nu_h > 0$ if and only of $h \in \mathcal{P}^{cc}$.

Bewley-model representation with the partition $\mathcal{P}$. Consider an agents with a history $h \in \mathcal{P}$ and a beginning-of-period wealth $a \in A$. Her budget constraint (18) can be expressed using policy functions as:

$$g_c\left(a, e_0(h)\right) + g_a\left(a, e_0(h)\right) = (1+r)a + w\left(e_0(h)\right).$$

Integrating this equality over $a \in A$ using the probability distribution $\Gamma^{\mathcal{P}}\left(a, h\right)$ yields after dividing by S_h:

$$c_h + a'_h = (1+r)\,\tilde{a}_h + w\left(e_0(h)\right). \tag{24}$$

Equation (24) can be interpreted as an average budget constraint for agents endowed with history h.

By construction of $\Gamma^{\mathcal{P}}$, we have for all $a \in A$, and $h' \in \mathcal{P}$, $\Gamma^{\mathcal{P}}(a, h') = \sum_{h\in\mathcal{P}} \Pi_{h,h'}\Gamma^{\mathcal{P}}(a, h)$. In other words, agent with history h transit at the next date to history h' with probability $\Pi_{h,h'}$.Note that transitions between some histories may be impossible and will therefore be assigned a zero probability. From equation (20), we deduce the following relationship between population sizes:

$$S_{h'} = \sum_{h\in\mathcal{P}} \Pi_{h,h'} S_h. \tag{25}$$

We can now express the average per-capita beginning-of-period wealth $\tilde{a}'_{h'}$ in the next period for agents with history $h' \in \mathcal{P}$:

$$\begin{aligned}
\tilde{a}'_{h'} &= \int_A a' \frac{\Gamma^{\mathcal{P}}\left(da', h'\right)}{S_{h'}}, \\
&= \sum_{h\in\mathcal{P}} \Pi_{h,h'} \int_A a' \frac{\Gamma^{\mathcal{P}}(da', h)}{S_{h'}} \frac{S_h}{S'_h}, \\
&= \sum_{h\in\mathcal{P}} \Pi_{h,h'} \int_A g_a(a, e_0(h)) \frac{\Gamma^{\mathcal{P}}\left(da, h\right)}{S_h} \frac{S_h}{S'_h}, \\
&= \sum_{h\in\mathcal{P}} \Pi_{h,h'} a'_h \frac{S_h}{S'_h},
\end{aligned} \tag{26}$$

In equation (26), the beginning-of-period wealth $\tilde{a}'_{h'}$ can be expressed as the weighted average of the end-of-period (in the previous period) wealth a'_h taking into account all possible previous histories h, the transition probabilities between histories, and the population size of for each history.

We can now turn to the expression of Euler equations for each histories $h \in \mathcal{P}$. The following proposition summarizes our result.

Proposition 1 (Allocation) *There exists a unique set of positive parameters $(\xi_h)_{h\in\mathcal{P}}$ such that for all $h \in \mathcal{P}$, the following equation holds:*

$$\xi_h u'(c_h) = \beta(1+r)\left(\sum_{h\in\mathcal{P}} \Pi_{h,h'}\xi_{h'}u'(c_{h'})\right) + \nu_h. \tag{27}$$

Proof. To prove the result, we switch to vectorial notation. We define $\boldsymbol{u}_\xi = (\xi_h u'(c_h))_{h\in\mathcal{P}}$, $\boldsymbol{\Pi} = (\Pi_{h,h'})_{h,h'\in\mathcal{P}}$, $\boldsymbol{\nu} = (\nu_h)_{h\in\mathcal{P}}$, and $\boldsymbol{I}$ the identity matrix of dimension equal to the cardinal of $\mathcal{P}$. The set of Equations (27) for all $h \in \mathcal{P}$ is equivalent to $\boldsymbol{u}_\xi(\boldsymbol{I} - \beta(1+r)\boldsymbol{\Pi}) = \boldsymbol{\nu}$. Assumption A guarantees that the vector $\boldsymbol{\nu}$ is not null. Furthermore, since $\beta(1+r) < 1$ and $\boldsymbol{\Pi}$ is a transition matrix, $\boldsymbol{I} - \beta(1+r)\boldsymbol{\Pi}$ is invertible. We deduce that $\boldsymbol{u}_\xi$ is uniquely defined as $(\boldsymbol{I} - \beta(1+r)\boldsymbol{\Pi})^{-1}\boldsymbol{\nu}$. We then obtain $(\xi_h)_{h\in\mathcal{P}}$ from $(u'(c_h))_{h\in\mathcal{P}}$ and $(\boldsymbol{u}_\xi)_{h\in\mathcal{P}}$. ■

We now define the following values for all $h \in \mathcal{P}$:

$$\eta_h \equiv \frac{\int_{-\bar{a}}^{\infty} u\left(g_c\left(a, e_0(h)\right)\right) d\Gamma^{\mathcal{P}}(a,h)}{u(c_h)}\frac{1}{S_h}, \tag{28}$$

$$\eta_h^{(1)} \equiv \frac{\int_{-\bar{a}}^{\infty} u'\left(g_c\left(a, e_0(h)\right)\right) d\Gamma^{\mathcal{P}}(a,h)}{u'(c_h)}\frac{1}{S_h}, \tag{29}$$

These quantities can be interpreted as measures of dispersion. The value η_h is a measure of the dispersion of utility of consumption among agents having the same history h. The quantities $(\eta_h)_{h\in\mathcal{P}}$ and $(\xi_h)_{h\in\mathcal{P}}$ are in general time-varying in presence of aggregate shocks.

Therefore, the intertemporal utilitarian welfare, which is $W = \sum_{h\in\mathcal{P}} \int_{-\bar{a}}^{\infty} \left(u\left(g_c\left(a, e_0(h)\right)\right) + \chi v(G)\right) d\Gamma^{\mathcal{P}}(a,h) + \beta W'$, can be expressed as:

$$W = \sum_{h\in\mathcal{P}} S_h \eta_h u(c_h) + \chi v(G) + \beta W'. \tag{30}$$

Summary of the Bewley model representation. The history representation of the Bewley model can be summarized by: (i) the budget constraint (24), (ii) the Euler equation (27), (iii) the wealth pooling equation (26), (iv) the intertemporal welfare (30), and (v) the financial market clearing equation

$$K = \sum_{h \in \mathcal{P}} S_h a_h. \tag{31}$$

This is a way to rewrite the Bewley model with richer information about agents' histories. This information is useless in the Bewley model because the current idiosyncratic state is a sufficient statistics for the behavior of agents (in addition to their beginning-of-period wealth). Writing the problem in the previous form is the first step for a finite-dimensional representation of the model in presence of aggregate shocks. Note that in the steady state with no aggregate shocks, the distributions are constant. As a consequence, the quantities $c_h, a_h, \tilde{a}_h, S_h$ are also constant.

3.3 Introducing aggregate shocks

An algorithm. We now present an algorithm that enables us to simulate the economy in presence of aggregate shocks but with exogenous taxes. Of note, there is no Ramsey problem to solve. The algorithm can be formalized as follows

1. Simulate a Bewley model to find the steady-state allocation ($A_t = 1$).
2. Choose a partition $\mathcal{P}$ and compute the projection the, $c_h, a_h, \tilde{a}_h, S_h$ and then ξ_h for $h \in \mathcal{P}$
3. Simulate the model with aggregate shocks, using a standard package like DYNARE.
4. Iterate over the partition $\mathcal{P}$, using a finer partition until second-order moments have converged, for a given accuracy.

Comparison with other methods. The previous algorithm is a refinement of algorithms that relies on perturbation techniques (Reiter (2009), Winberry (2016), Bhandari, Evans, Golosov, and Sargent (2016) or Ahn, Kaplan, Moll, Winberry, and Wolf (2017)). Indeed, as we do not specify the partition $\mathcal{P}$, it includes the algorithm of Reiter (2009), as will become clear in Section 5. The key difference with other methods is that we introduce the coefficients $(\xi_h)_{h \in \mathcal{P}}$ that enable to properly account for history aggregation in Euler equations. Indeed, the coefficients $(\xi_h)_{h \in \mathcal{P}}$ are a relevant measure of the heterogeneity in Euler equations for a given history $h \in \mathcal{P}$. Our algorithm therefore takes advantage of information about the steady state in the Bewley model that was previously discarded with other perturbation methods.

4 Ramsey program

4.1 Optimal policies

We now derive optimal Ramsey policies in the Bewley model. Comparing the Ramsey allocations in our setup with those of a complete insurance-market economy will enable us to identify the specific role of redistribution and the lack of insurance. However, solving for Ramsey policies in the general case is difficult. Indeed, one has to introduce additional state variables, such as the distribution of Lagrange multipliers for the relevant individual constraint.[1] Solving for this joint distribution is particularly difficult.

The main idea of the current method is to solve for the Ramsey optimal policy for the model projected on the partition $\mathcal{P}$. In a nutshell, we solve for the exact solution of an approximated model, whereas other methods provide approximate solutions of the exact model. We first explain the methodology to solve the model projected on $\mathcal{P}$, we then describe our algorithm for computing Ramsey policies and we finally discuss the relationships with other methods.

The Ramsey problem consists in determining the fiscal policy –here equivalently, public spending G_t and post-tax wage rate w_t– that corresponds to the "best" competitive equilibrium, according to an aggregate welfare criterion. In other words, the planner has to select fiscal policy and individual choices, subject to government and individual budget constraints (33) and (34), and subject to Euler equations (35) –that guarantee the optimality of individual choices.

[1]The relevant individual constraint depends on the way the Ramsey problem is written. As we discuss below, in the Lagrangian approach of Marcet and Marimon (2011), these relevant constraints are the individual Euler equations. Bhandari, Evans, Golosov, and Sargent (2016) use a primal approach and thus consider the individual Lagrange multiplier on the budget constraint.

Formally, the Ramsey problem can be written as follows:

$$\max_{\left(r_t, w_t, G_t, (a_t^i, c_t^i)\right)_{t\geq 0}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\sum_{h \in \mathcal{P}} S_{h,t} \eta_h u\left(c_{h,t}\right) + v\left(G_t\right) \right) \right], \tag{32}$$

$$G_t + K_{t-1} r_t + L_t w_t \leq K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1}, \tag{33}$$

for all $h \in \mathcal{P}$:

$$c_{h,t} + a_{h,t} \leq \left(1 + r_t\right) \tilde{a}_{h,t} + w_t\left(h\right), \tag{34}$$

$$\xi_h u'\left(c_{h,t}^b\right) = \beta(1 + r_t) \mathbb{E}_{h'}\left(\xi_{h'} u'\left(c_{h',t+1}\right)\right) + \nu_h, \tag{35}$$

$$\tilde{a}_{h,t+1} = \sum_{h' \succeq g} \Pi_{g,h,t} a'_{g,t} \frac{S_{g,t}}{S_{h,t+1}}, \tag{36}$$

$$K_t = \sum_{h \in \mathcal{P}} S_{h,t} a_{h,t}, \;\; L_t = \left(1 - s_t\right) L_{t-1} + f_t \left(1 - L_t\right), \tag{37}$$

$$c_{h,t}^i, (a_{h,t}^i + \overline{a}) \geq 0. \tag{38}$$

Other constraints are the pooling equation (36) that comes from history representation, aggregation and positivity constraints (37) and (38).

The key assumption to solve this model is that the quantities $(\eta_h)_{h\in\mathcal{P}}$ and $(\xi_h)_{h\in\mathcal{P}}$ are not time-varying and set to their steady-state optimal values. As the set of equations is finite, it is easy to derive first-order conditions. We discuss in Section 4.2 below issues related to second-order conditions. First, we denote by $\beta^t \lambda_{h,t}$ the discounted Lagrangian multiplier of the Euler condition (34) for history h, and by $\beta^t \mu_t$ the Lagrangian multiplier on the government budget constraint (33). The Lagrange multiplier $\lambda_{h,t}$ measures how costly it is for the planner to internalize the Euler equation. To ease the interpretation of first-order conditions, we introduce the following notation:

$$\Lambda_{h,t} \equiv \frac{\sum_{h' \in \mathcal{P}} S_{h',t-1} \lambda_{h',t-1} \Pi_{h',h,t}}{S_{h,t}}, \tag{39}$$

The variable $\Lambda_{h,t}$ is the average per capita cost of internalizing the previous period Euler equation for agents with history h today. Roughly speaking, this is the past average of past values of Lagrange multiplier $(\lambda_{h,t-1})_{h\in\mathcal{P}}$. We also define the quantity $\psi_{t,h}$ as

$$\psi_{t,h} \equiv \eta_h u'(c_{t,h}) + \xi_h \left((1 + r_t)\Lambda_{t,h} - \lambda_{t,h}\right) u''(c_{t,h}), \tag{40}$$

which is the *social valuation of liquidity of agents h*. Indeed, if all agents h receive one additional unit of goods today, this additional unit will be valued $\eta_h u'(c_{t,h})$. This value only accounts for private valuation, but should also include the effect on the internalization cost of Euler equations.

Indeed, this additional unit affects agents' incentive to save from period $t-1$ to period t and from period t to period $t+1$. This effect is captured by the second term at the right hand side, proportional to $u''(c_{t,h})$.

We now provide the expressions of the first order conditions of the Ramsey program. Of note, as we discuss in Section 4.2, these conditions are necessary, but not sufficient, to guarantee the existence of an internal solution.

$$\mu_t = v'(G_t), \tag{41}$$

$$\psi_{t,h} = \beta \mathbb{E}_t \left((1+r_{t+1}) \sum_{h' \in \mathcal{P}} \Pi_{t,h,h'} \psi_{t+1,h'} \right), \text{ for } h \notin \mathcal{P}^{cc}, \tag{42}$$

$$\mu_t L_t = \sum_{h \in \mathcal{P}} S_{h,t} \left(\phi 1_{e_0(h)=1} + (1-\tau_t) 1_{e_0(h)=0} \right) \psi_{t,e^N}, \tag{43}$$

where we recall that $e_0(h)$ denotes the current employment status of agent with history h.

4.2 Remark on the convexity of the program

A traditional problem with Ramsey program is that the set of feasible allocations is not convex in general. This problem is quite general and also exists in a representative-agent economy. The nonconvexity is precisely related to constraint associated to Euler equation – which is neither convex nor linear. Therefore, if first-order conditions are still necessary, they may be non-sufficient and generate three different types of problems:

1. the first order condition may characterize a local minimum;
2. the steady-state solution may not exist;
3. multiple equilibria may exist.

The first concern can be easily addressed, for instance by checking that small variations around the solution allocation do not yield a higher aggregate welfare. The second concern has been raised by Straub and Werning (2014), who show that in some cases the solution of the planner may not be an interior solution with constant real variables.[2] The possibility to solve the model with perturbation methods helps solve this issue. Indeed, studying the behavior of the model after perturbing the steady state with small aggregate shocks, provides insight regarding the

[2] Recent contributions such as Chari, Nicolini, and Teles (2016) show that the behavior of Lagrange multipliers depends on the set of instruments available to the planner. In addition, Chen, Chien, and Yang (2017) show theoretically that in an incomplete insurance-market model that the solution is interior.

subsequence convergence –or not– toward the interior solution. The last concern is more difficult to properly address. Up to our knowledge, the only imperfect solution consist sin exploring the convergence for various initial values and checking that the local maximum is indeed a global one.

4.3 The algorithm

We now present the algorithm for computing the steady-state. The algorithm consists of two steps. First, we find the interior steady state, and second we rely on perturbation methods to investigate the dynamic behavior.

Steady state. The steps of the algorithm for the steady state are as follows.

1. Choose a partition $\mathcal{P}$.
2. Assume that an initial element h^{cc} gathers all credit constraints agents.
3. Choose initial values for the interest rate r and the post-tax wage rate w.
   (a) Solve the full Bewley model with r and w. Deduce the aggregate quantities K, Y and G. Project the solution on $\mathcal{P}$ to compute the history-dependent variables c_h, a_h, $\tilde{a}_h$, S_h, η_h, and ξ_h.
   (b) Set a value for μ.
      i. Choose the values $(\psi_{h^{cc}})_{h^{cc} \in \mathcal{P}^{cc}}$ (for credit-constrained agents). Solve for ψ_h, for all $h \notin \mathcal{P}^{cc}$ (unconstrained histories) using equation (42). Using (39) and (40), deduce $(\lambda_h)_{h \in \mathcal{P}}$. Iterate on $(\psi_{h^{cc}})_{h^{cc} \in \mathcal{P}^{cc}}$ until $\lambda_{h^{cc}} = 0$ for all $h^{cc} \in \mathcal{P}^{cc}$.
      ii. Iterate on μ until equation (41) holds.
   (c) Iterate on r and w until the equation (43) is fulfilled and until financial market clears.

The important step of the algorithm is the step 3. (a). Solving the Bewley model enables us to extract all relevant steady state information for the Ramsey model. We can do so because the program of households only depends on post-tax prices.

Dynamics. The steps for computing the dynamics are the following ones.

1. If needed, project the steady-state allocation and equations on a coarser partition.

2. Gather all non-linear equations (a finite number) and solve the model using perturbation methods for the aggregate shocks at the desired order. The last part can easily be done using a standard package, such as Dynare for instance.

Our solution method makes the simulation of the model with aggregate shocks the easy part when solving the Ramsey program.

4.4 Comparison with other methods

To our knowledge, only three other papers provide general solution method to derive optimal Ramsey policies in incomplete insurance-market models.

First, Açikgöz (2015) provides an algorithm to solve for the steady-state allocation of the Ramsey program. He assumes some specific functional forms and show the convergence of the algorithm. This is a way to find the joint distribution over Lagrange multipliers and initial asset holdings. At this stage, we are not aware of any application of this algorithm to an economy with aggregate shocks.

Second, Nuño and Moll (2017) use a continuous-time approach and mean-field games to characterize optimal steady-state allocations. Their algorithm develops a projection method to characterize the relevant value functions and Lagrange multipliers. Our solution makes a more extensive use of the steady-state properties of the Bewley model, that enables us to properly distort the projection on a relevant grid. Although our model is expressed in discrete time, a methodology similar to ours can be applied to continuous-time models. An additional gain of our method in discrete time is that introducing aggregate shocks is straightforward, as we have seen above.

Third, Bhandari, Evans, Golosov, and Sargent (2016) present a solution method for models with aggregate shocks. Their solution relies on perturbation methods around time-varying allocations (and not around the steady-state). They solve the model by approximating the actual distribution by 100,000 agents. As we use more extensively the steady-state properties of the Bewley model, we can simulate the economies with a very small number of agents –see Section 6. As a consequence, our solution allows us to study Ramsey problems with a number of instruments.

5 Implicit partitions

We now explain how to apply our framework to an implicit partition. We start with describing how to derive an implicit partition in the space of history based on the steady-state distribution of wealth in the Bewley model. We then discuss the relationship with Reiter (2009).

As explained in Section 2.5.2, the resolution of the Bewley model generates a steady-state wealth distribution, whose cumulated distribution function is denoted $(a, e) \mapsto \Gamma(a, e)$, where a belongs to the compact set $[-\bar{a}, a^{max}]$ and $e \in \{0, 1\}$. We now consider a finite partition $(B_h)_{h=1,\ldots,H}$ of the wealth set $[-\bar{a}, a^{max}]$ with $H > 0$. The partition elements must verify the following properties:

$$\begin{cases} [-\bar{a}, a^{max}] = \cup_{h=1,\ldots,H} B_h, & \\ B_h \cap B_{h'} = \emptyset & \text{for all } h \neq h'. \end{cases}$$

We will distinguish agents by their wealth holdings and their current employment status. We therefore consider $2H$ agents' types. A type $h = 1, \ldots, H$ refers to an unemployed agent whose beginning-of-period wealth belongs to the set B_h. Conversely, a type $h = H + 1, \ldots, 2H$ refers to an employed agent whose beginning-of-period wealth belongs to the set B_{h-H}. Using the steady-state distribution of the Bewley model, we can compute for any agent of type $h = 1, \ldots, 2H$, the average steady-state consumption c_h, the average beginning-of-period wealth $\tilde{a}_h$, the average end-of-period wealth a'_h, as well as the measure S_h of agents of the same type –with $\sum_{h=1}^{2H} S_h = 1$. Furthermore, we can also compute, using the steady-state decision rules, the measures $(F_{h,h'})_{h,h'=1,\ldots,2H}$ of agents transiting from a type h in the previous period to a type h' in the current period. From these measures, we can construct the transition matrix $(\Pi_{h,h'})_{h,h'=1,\ldots,2H}$, where the probability $\Pi_{h,h'}$ to transit from state h to state h' is expressed as:

$$\Pi_{h,h'} = \frac{F_{h,h'}}{S_h}.$$

Importantly, this partition in the space of wealth implicitly implies a partition in the space of histories. Indeed, a given type $h = 1, \ldots, 2H$ implicitly corresponds to all histories that generate a beginning-of-period wealth B_h if unemployed (and $h \leq H$) or B_{h-H} if employed (and $h > H$). Although we do not know these histories explicitly, this partition in the space of histories implicitly exists. As a consequence, the apparatus of Section 3 can be used to construct the quantities $(\xi_h)_{h=1,\ldots,2H}$ and $(\eta_h)_{h=1,\ldots,2H}$. Once these quantities have been constructed, we can follow Section 4.1 to build a reduced-history representation of the model that enables to compute optimal policies.

Compared to Reiter (2009) initial algorithm, the main computational difference with our approach is that we capture heterogeneity in Euler equations using the coefficients $(\xi_h)_{h=1,...,2H}$. As shown below, this allows us to study the dynamics with a small number of agents. This deviation also permits the simulation of optimal policies with the same techniques. A last gain compared to Reiter's algorithm is the easiness to solve the model with time-varying idiosyncratic risk.[3]

The model structure implies that transition probabilities Π and the size $(S_h)_{h=1,...,2H}$ of partition elements are constant, even outside of the steady-state, when idiosyncratic risk is not time-varying (i.e., the matrix M is constant). This result holds even in presence of other aggregate shocks, such as aggregate TFP shocks affecting the wealth of agents. This non-intuitive results is better understood when thinking about the difference between the partition in the space of wealth and the implicit partition in the space of histories. If the reasoning is solely based on the partition in the space of wealth, then any aggregate shock affecting wealth (such as a TFP shock) should affect saving decisions and thus the measure of agents in a given wealth bracket B_h, even if idiosyncratic risk is constant. This is not the case anymore if the reasoning is based on the set of idiosyncratic histories. Indeed, this set is not affected by aggregate shocks –because the transition matrix M is constant– which makes it possible to identify this set in the steady-state. We provide in Appendix B the sketch of the algorithm to solve models with time-varying idiosyncratic risk.

6 Numerical examples

We now provide two numerical examples to apply the previous setup: one with exogenous taxes and another one with optimal fiscal policy. These examples are chosen to be very simple, so as to present in a very transparent framework the properties of the proposed methodology. In the first case with exogenous taxes, we compute the dynamics of the model with TFP shocks and compare the current algorithm with alternative ones, such as Reiter (2009) or Krusell and Smith (1998). In the second case, we solve for the optimal provision of the public good and time-varying capital taxes after technology shocks.

[3]Reiter (2009) considers constant i.i.d. idiosyncratic risk, with a continuous support. Winberry (2016) assumes an idiosyncratic risk, where the transition matrix M (defined in equation (2)) is not time-varying.

Parameter	Description	Value
β	Discount Factor	0.96
α	Capital share	0.36
δ	Depreciation rate	0.1
b	UI replacement rate	0.1
π_{01}	U to E probability	0.5
π_{10}	E to U probability	0.038
ρ^A	autocorr. TFP shock	0.859
σ^A	Std dev. innov. TFP	0.014
χ	Pref. pub good	0

Table 1: Parameter values

6.1 Parameter values

Table 1 provides the steady-state parameter values which are common in the three economies. These parameters are standard for annual parametrization. Most of the parameters are taken from Den Haan (2010). They are used by Winberry (2016) to compare model outcome with different computational methods. We consider here the case where the preference for the public good is 0, such that $G_t = \tau_t^K = 0$. As a consequence, we solve for the standard model with TFP shock and without taxes on capital.

6.2 Model with exogenous taxes

6.2.1 Steady-state comparison

We first provide steady-state comparison for different values of H. Note that in each economy parametrized by H, the capital stock, the real wage and the real interest rate are exactly the same as in the true Bewley model, by construction.

Table 2 summarizes the comparison between the economies for different values of H. More precisely, we report in this table for different values of H, the cross-sectional standard deviation of $(\xi_h)_{h=1,\ldots,2H}$, as well as the Gini coefficient in the economy. There is one exception, with is the last column with $H = 17$, for which we impose $\xi_h = 1$ for all h and that corresponds to a null standard deviation of ξ by construction. Table 2 includes three lines. The first one provides

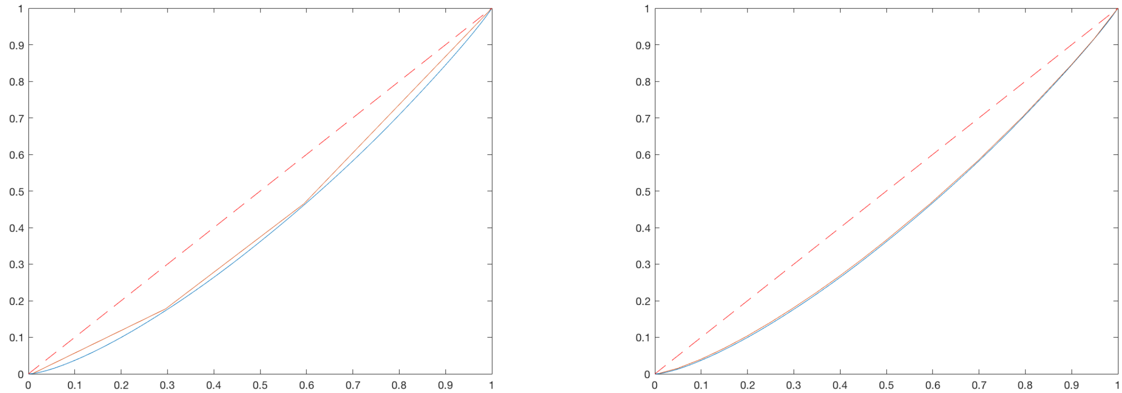
H	4	11	**17**	32	100	*17*
Gini(%)	17	19	19	19	19	23
$sd\,(\xi)(\%)$	5.05	2.43	1.54	0.95	0.79	0

Table 2: Steady-state comparisons

the number H , implying H employed agents and H unemployed ones. For each value of H, we have chosen a partition with roughly equidistant agents in the wealth distribution. For instance, in the case where $H = 17$ (which will be our benchmark case), the partition of the wealth distribution is $[8.10^{-2}, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 70, 80, 90, 95]$ (in percentage points of the saving space $[-\bar{a}, a^{max}]$). Our partition therefore involves agents with wealth between 0% and 8.10^{-2}% of the maximal holding value a^{max} (since the lower bound $\bar{a}$ has been set to 0), agents with wealth between 8.10^{-2}% and 5% of a^{max} and so on and so forth. The initial segment $[0\%, 8.10^{-2}\%]$ corresponds to credit constrained unemployed households in the Bewley model.

The second line of the Table 2 contains the Gini coefficient of the wealth distribution. We can observe that the Gini coefficients rapidly converge toward its steady state value 19. In the last line of Table 2, we report the standard deviation of ξ across agents. One can show that this standard deviation converges toward 0 when H increases. This convergence means that the finer the partition, the smaller the inter-element heterogeneity. For our construction, it is not a problem that some ξ_h differ from 1, since it is precisely a correction that we introduce in the Euler equation to account for wealth heterogeneity within partition elements. Finally, in the last column labeled *17* corresponds to the same economy as the benchmark case ($H = 17$), except that we impose $\xi_h = 1$ for all agents h. This steady-state distribution of this economy is very different from the one in the Bewley economy. For instance, the Gini coefficient amount to 23, well above the actual Bewley value of 19. As a consequence, this shows that the $(\xi_h)_{h=1,\ldots,2H}$ are useful to match both the cross-section and the dynamics of the Bewley model, with a small number of agents.

The fact that a small number of agents allows our model to reproduce the correct wealth distribution stems from the fact that our construction in fact approximates the Lorenz curve by H affine functions. As can be seen in Figure 1, four segments (panel 1a) provide a first approximation, while seventeen segments ($H = 17$ on panel 1b) do a very good job in reproducing



(a) Case $H = 4$

(b) Case $H = 17$ (benchmark)

Figure 1: Lorenz curve for two economies

the shape of the Lorenz curve. On each panel of Figure 1, we plot the Lorenz curve for the Bewley model (blue line) and for the approximated model (red line). The dashed red line represents the 45-degree line. On the panel 1b for $H = 17$, the Lorenz curves of the Bewley and the approximated models are almost indistinguishable.

6.2.2 Comparison of business-cycle statistics

We now report the outcomes of the model in terms of second-order moment. We compare the outcomes of our solution technique to those of two other standard simulation techniques: Krusell and Smith (1998) and Reiter (2009) with Winberry (2016) refinement. We report second-order moments in Table 3 for five key variables: output, consumption, investment, real wage, and real interest rate. For each variable, we report its standard deviation normalized by the standard deviation of output (except for output). We also report the correlation of the variable with output.

The moments generated by the three methods are very similar to each other. As a consequence, a small number of agents is sufficient for reproducing the cross-sectional heterogeneity and the model dynamics when the inter-group heterogeneity is captured with $(\xi_h)_h$.

Variable	Sd (relative to Sd(Y))			Correlation with Y		
	Model	KS	Reiter	Model	KS	Reiter
Output (%, base)	1.32	1.32	1.32	1	1	1
Consumption	0.49	0.49	0.50	0.91	0.91	0.92
Investment	2.64	2.67	2.64	0.98	0.98	0.98
Real wage	1	1	1	1	1	1
Real interest rate	0.15	0.15	0.15	0.90	0.90	0.90

Table 3: Business cycle statistics

H	358	394	976
G	0.088	0.091	0.091
τ^L	0.0965	0.100	0.100
Sd(ξ)	0.015	0.014	0.014
Sd(η)	0.003	0.003	0.003

Table 4: Steady-state convergence

6.3 The model with optimal policies

6.3.1 Steady-state results

We now consider an economy where agents derive utility from the consumption of the public good and where labor taxes are adjusted to maximize welfare. The parameters are the same as in Table 1, except for the the parameter χ which is set to a value $\chi = 0.082$, and the function v is $v(G) = \log(G)$. As discussed below, this value for χ is set to imply a realistic labor tax.

We solve for the optimal steady state provision of public goods G and labor tax τ^L, for different partition sizes. We use implicit partitions using the distribution of wealth, as explained in Section 5. In each partition, there is the same measure of agents in each element of the partition, except for the initial elements, which gather credit-constrained agents. Results are gathered in Table 4. These results illustrate how the fiscal system converges when the partition becomes finner and show that our history representation provides a good approximation of the initial problem.

The steady-state values for the public good G and the labor tax τ^L converge to the values

Variable	Sd (relative to Sd(Y))			Correlation with Y		
Economy	CM	$H=4$	$H=22$	CM	$H=4$	$H=22$
output (%, base)	1.32	1.33	1.33	1	1	1
Consumption	0.50	0.50	0.50	0.91	0.92	0.92
Investment	2.67	2.74	2.79	0.98	0.97	0.97
Public spend. G	0.05	0.05	0.05	0.91	-0.18	-0.12
Real wage	1.05	1.12	1.12	1	1	1
Int. rate (bef. tax)	0.15	0.15	0.15	0.90	0.89	0.89
Labor tax	0.05	0.12	0.12	-0.94	-0.91	-0.91

Table 5: Business cycle statistics of Ramsey allocation

of 9.1% and 10% respectively. These values are almost reached with a partition of 394 elements, that will be our benchmark. The standard deviations $sd(\xi)$ and $sd(\eta)$ –corresponding to $(\xi_h)_{h=1,\ldots,2H}$ and $(\eta_h)_{h=1,\ldots,2H}$– are small and decreasing, while the empirical means $E_h\xi_h$ and $E_h\eta_h$ almost exactly amount to 1. The partition $H = 394$ will be used as our benchmark and we will apply perturbation methods to this partition.

6.3.2 Comparison of business-cycle statistics

We now compute business cycle statistics to quantify the effects of aggregate technology shocks. We reports the results for three economies. The first one is the complete market economy (henceforth, CM). This complete market economy can be seen as corresponding to a partition with $H = 1$ –there is no idiosyncratic risk. As taxes are non-distorting, the complete-market allocation is also the first best-allocation. The second economy corresponds to a partition with a small number of elements, $H = 4$. The third economy is an economy with a partition involving $H = 22$ elements. Note that for the last two economies, we project the previous steady-state partition with $H = 394$ on two coarser partitions. The coarser partition is indeed sufficient for generating an acceptable approximate dynamics. This interesting result does not come from the fact that the CM and incomplete market economies generate similar outcomes, as will be shown in Figure 2 plotting impulse response functions. We report the second-order moments of the three economies in Table 5.

First, second-order moments are very similar in incomplete market economies with $H = 4$ and $H = 22$. Second, the moments are also roughly similar in the CM economy and in the other

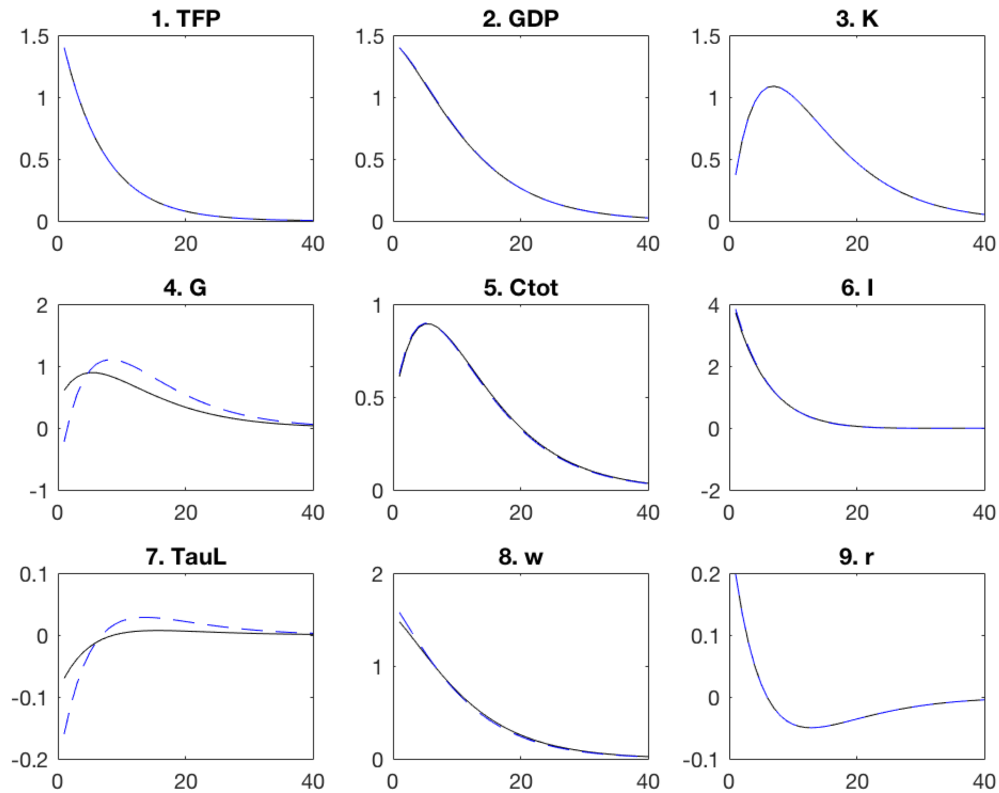


Figure 2: Impulse response functions

two incomplete-market economies, except for the correlation of public spending with output. The latter is positive and close to one in the CM economy, while it is negative for both $H = 4$ and $H = 22$. The difference in this correlation comes from the redistributive effects of labor tax in the incomplete market economy.

To better understand this difference, we also plot the IRFs after a technology shock in Figure 2. All variables are reported as relative deviations from steady state values, except for the interest rate r and the labor tax τ^L which are reported in absolute deviations from steady state values. In Figure 2, the first panel reports the TFP, while other panels plot the IRFs for key variables. For each variable, we plot the results of the CM economy in plain line, while those of the incomplete-market economy (with $H = 22$, IM henceforth) are plotted with dashed line. Similarly to second-order moments, all IRFs, but the one of public spending (1st column, 2nd row) and the labor tax (1st column, last row) are very similar. After a positive TFP shock, labor taxes decrease in both economies, but the decrease is steeper in the incomplete-market economy, because decreasing labor tax enables the planner to transfer resources to employed agents. Due to these lower resources, the provision of public goods diminishes on impact in the IM economy. This explains the different signs of the correlation between output and public spending in CM and IM economies.

To sum it up, while a relatively fine partition is needed to compute the steady-state with optimal taxes, the dynamics can be simulated on coarser grid. Even though the setup is relatively simple, CM and IM economies generate significant differences in optimal public policies.

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Appendix

A Derivation of the History-Representation of the First-Order conditions

$$\max_{\left(r_t,w_t,G_t,(a_t^i,c_t^i)\right)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left(\sum_{h\in\mathcal{P}} S_h\eta_h u\left(c_h\right)+v\left(G_t\right)\right)\right], \tag{44}$$

$$G_t+K_{t-1}r_t+L_tw_t\leq K_{t-1}^{\alpha}L_t^{1-\alpha}-\delta K_{t-1} \tag{45}$$

for all $h\in\mathcal{P}$:

$$c_{h,t}+a_{h,t}\leq\left(1+r_t\right)\tilde{a}_{h,t}+w_t\left(h\right) \tag{46}$$

$$\xi_h u'\left(c_{h,t}^b\right)=\beta(1+r_t)\mathbb{E}_{h'}\left(\xi_{h'}u'\left(c_{h',t+1}\right)\right)+\nu_h \tag{47}$$

$$\tilde{a}_{h,t+1}=\sum_{h'\succeq g}\Pi_{g,h,t}a'_{g,t}\frac{S_{g,t}}{S_{h,t+1}} \tag{48}$$

$$K_t=\sum_{h\in\mathcal{P}} S_{h,t}a_{h,t}\Lambda_t,\ \ L_t=\left(1-s_t\right)L_{t-1}+f_t\left(1-L_t\right), \tag{49}$$

$$c_{h,t}^i,\left(a_{h,t}^i+\overline{a}\right)\geq 0, \tag{50}$$

The Lagrangian can be written as

$$\begin{aligned} J=\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{h\in\mathcal{P}}S_{h,t}\eta_h u(c_{h,t})-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{h\in\mathcal{P}}S_{h,t}\lambda_{h,t} \\ \times\left(\xi_h u_c(c_{h,t})-\nu_{h,t}-\beta\mathbb{E}_t\left[\sum_{h'\in\mathcal{P}}\Pi_{h,h',t+1}\xi_{h'}u_c(c_{h',t+1})(1+r_{t+1})\right]\right)\end{aligned} \tag{51}$$

Define

$$\Lambda_{h,t}=\frac{\sum_{h'\in\mathcal{P}}S_{h',t-1}\lambda_{h',t-1}\Pi_{h',h,t}}{S_{h,t}}, \tag{52}$$

Hence

$$\begin{aligned}\mathcal{L}=\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{h\in\mathcal{P}}S_{h,t}\left(\eta_h u(c_{h,t})+\xi_h u_c(c_{h,t})\left(\Lambda_{h,t}(1+r_t)-\lambda_{h,t}\right)\right) \\ -\mathbb{E}_0\sum_{t=0}^{\infty}\mu_t\beta^t\left(K_{t-1}^{\alpha}L_t^{1-\alpha}-\delta K_{t-1}-G_t-K_{t-1}r_t-L_tw_t\right),\end{aligned} \tag{53}$$

with

$$\begin{aligned} c_{h,t} + a_{h,t} &\leq (1 + r_t)\, \tilde{a}_{h,t} + w_t\,(h) \\ \tilde{a}_{h,t+1} &= \sum_{h' \succeq g} \Pi_{g,h,t} a'_{g,t} \frac{S_{g,t}}{S_{h,t+1}} \end{aligned}$$

Derivative with respect to w_t.

We obtain:

$$\mu_t L_t = \sum_{h \in \mathcal{P}} S_{h,t} \frac{w_{h,t}}{w_t} \psi_{t,e^N}. \tag{54}$$

Then the first-order conditions of the Ramsey program can be written as

$$\begin{aligned} \mu_t &= v'\,(G_t) \\ \psi_{t,h} &= \beta \mathbb{E}_t \left((1 + r_{t+1}) \sum_{h' \in \mathcal{P}} \Pi_{t,h,h'} \psi_{t+1,h'} \right), \text{for } h \neq h_{cc} \\ \mu_t L_t &= \sum_{h \in \mathcal{P}} S_{h,t} \frac{w_{h,t}}{w_t} \psi_{t,e^N}. \end{aligned}$$

Note that $\frac{w_{h,t}}{w_t} = \phi$ if the agents is unemployed and $\frac{w_{h,t}}{w_t} = 1 - \tau_t$ is the agent is employed.

B Algorithm for time-varying idiosyncratic risk

We provide an algorithm to introduce time-varying idiosyncratic shocks. To simplify the exposition, we assume that in the steady-state, any agent's type h moves to either the previous, the same or the next set of histories, i.e. any type h agent becomes either $h-1$, h, $h+1$, or $h-1 \pm H$, $h \pm H$, $h+1 \pm H$.[4] We write this assumption more formally.

Assumption B *We assume that:* $F_{h,h'} > 0$ *only if* $h' = h-1, h, h+1$ *modulo* H.

From the knowledge of saving policies in the Bewley economies (Huggett 1993 for instance), we know that unemployed agents dis-save (hence they go from h to either h or $h-1$ modulo H) and employed agents save (they go from h to either h or $h+1$ modulo H). We denote ϕ_h the fraction of agents of a given type h, who remain of type h (modulo H) in the next period . Correspondingly, a fraction $1 - \phi_h$ of type-h agents become of another type, either $h-1$ or $h+1$ (modulo H). We therefore deduce

[4]We have to consider the results modulo H because unemployed agents an become employed and vice et versa.

$$
\begin{aligned}
F_{t,h,h} &= \phi_h S_{t,h} M_t(0,0) && \text{if } h = 1, \dots, H \\
F_{t,h,h-1} &= (1-\phi_h) S_{t,h} M_t(0,0) && \text{if } h = 2, \dots, H \\
F_{t,h,h+H} &= \phi_h S_{t,h} M_t(0,1) && \text{if } h = 1, \dots, H \\
F_{t,h,h-1+H} &= (1-\phi_h) S_{t,h} M_t(0,1) && \text{if } h = 1, \dots, H \\
F_{t,h,h} &= \phi_h S_{t,h} M_t(1,1) && \text{if } h = H+1, \dots, 2H \\
F_{t,h,h+1} &= (1-\phi_h) S_{t,h} M_t(1,1) && \text{if } h = H+1, \dots, 2H-1 \\
F_{t,h,h-H} &= \phi_h S_{t,h} M_t(1,0) && \text{if } h = H+1, \dots, 2H \\
F_{t,h,h+1-H} &= (1-\phi_h) S_{t,h} M_t(1,0) && \text{if } h = H+1, \dots, 2H-1
\end{aligned}
$$

with $\phi_1 = \phi_{2H} = 1$. The values for ϕ_h $(h = 1, \dots, 2H)$ can be derived from the steady-state model, for which all $F_{h,h'}$ are known. We can then deduce the time-varying fractions $(F_{t,h,h'})$ from the matrix M_t, and finally the time-varying transition probabilities: $\Pi_{t,h,h'} = F_{t,h,h'}/S_{t,h}$. This then enables us to solve the model with time-varying idiosyncratic risk.