Comparative Valuation Dynamics in Models with Financing Restrictions*

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June 27, 2018

Abstract

This paper develops a theoretical framework to nest many recent dynamic stochastic general equilibrium economies with financial frictions into one common generic model. Our goal is to study the macroeconomic and asset pricing properties of this class of models, identify common features and highlight areas where these models depart from each other. In order to characterize the asset pricing implications of this family of models, we study their term structure of risk prices and risk exposures, the natural extension of impulse response functions in economic environments exhibiting non-linear behaviors.

*First draft February 2018. We thank Joseph Huang for excellent research assistance, and Amy Boonstra for her unconditional support. We would like to gratefully acknowledge the Macro-Finance Modeling initiative for their generous financial support. For their feedback, we thank the conference participants at the 2nd Research Conference of the MMCN, seminar attendees at Northwestern University, and participants at the Economic Dynamics Working Group at the University of Chicago.

1 Introduction

In this paper, we develop a theoretical framework and diagnostic tools for comparing and contrasting dynamic macroeconomic models with financial frictions. The literature studying this class of models has expanded considerably following the 2008 financial crisis, as researchers are trying to understand how financial intermediaries facing different types of regulatory and contractual constraints influence macroeconomic outcomes.[1] To what extent do the existing models differ in their macroeconomic and asset pricing implications? To what extent are they similar? Those questions are the key drivers motivating our research and prompting us to undertake a comparison exercise.

To perform comparisons in a systematic way, we pose a theoretical continuous-time dynamic stochastic general equilibrium model that nests several benchmark models from the literature. We focus on a production economy with two types of agents having different productivities, preferences, and financial constraints. We follow previous research by introducing a productive role for the financing of investments by one class of agents meant to capture financial intermediaries and their managers. These agents are more productive and more risk tolerant than households, but they also face financing restrictions. These restrictions include limits on short-sales, equity issuance, and leverage. The occasionally binding nature of those constraints generate an important source of nonlinearity in our model. We allow both agent types to hedge aggregate risk exposures in financial markets, but subject to constraints. Finally, we augment standard productivity shocks with growth-rate shocks, aggregate volatility shocks, and idiosyncratic volatility shocks.

This set of assumptions results in an economic environment that nests multiple models within a common setup to support model comparisons. Although this general nesting model is less analytically tractable and more computationally complex than any of the benchmark

[1]Beginning from the seminal "financial accelerator" papers of Kiyotaki and Moore (1997) and Bernanke, Gertler, and Gilchrist (1999), there has been explosive growth in the literature on macroeconomic dynamics under financial frictions. See Brunnermeier and Sannikov (2014), He and Krishnamurthy (2012, 2013, 2014), Adrian and Boyarchenko (2012), Moreira and Savov (2017), Phelan (2016), Di Tella (2017), and Klimenko et al. (2016) for some of the core issues in continuous-time models. An even more developed asset pricing literature that predates the continous-time macro literature explores similar frictions in endowment economies: e.g., Basak and Cuoco (1998), Basak and Croitoru (2000), Basak and Shapiro (2001), Gromb and Vayanos (2002), Kondor (2009), Garleanu and Pedersen (2011). This set of macro-finance models has been extended to examine conventional monetary theory (Drechsler, Savov, and Schnabl (2018), Brunnermeier and Sannikov (2016a), Di Tella and Kurlat (2017)), unconventional monetary policy (Silva (2016)), macro-prudential policies (Di Tella (2016), Caballero and Simsek (2017)), international capital flows (Brunnermeier and Sannikov (2015)), and the cross-section of asset prices (Dou (2016)). For the recent wave of related discrete-time models, see Gertler and Karadi (2011), Gertler and Kiyotaki (2010), Mendoza (2010), Bianchi (2011), Gertler and Kiyotaki (2015), Christiano, Motto, and Rostagno (2014). For empirical work linking financial intermediary leverage and net worth to asset prices and macroeconomic conditions, see Adrian and Shin (2010), Adrian and Shin (2013), Adrian, Etula, and Muir (2014), He, Kelly, and Manela (2017), Muir (2017), Siriwardane (2016).

models from the literature, our nesting framework offers two important advantages. First, performing comparisons across models is complicated by the fact that each model has its own special auxiliary assumptions and its own calibration. By nesting several models, we can hold fixed auxiliary assumptions and parameters, using only a single parameter to transition from one model to another. This simplifies the comparison exercise. Second, nesting different models necessarily introduces interactions between their assumptions. For example, by nesting models "A" and "B", our model allows us to solve a version of model "A" with some of the features from model "B", say its shock or preference structure. These interactions help us distinguish between various assumptions on financial frictions, and allow us to uncover new mechanisms not previously explored by previous articles in this literature.

To perform comparisons, we focus our attention on the asset-pricing implications of our model. Like most of the literature, we characterize asset prices by the model's stochastic discount factor (SDF). Unlike complete markets' environments, our model features two SDFs, arising from the different investment opportunity sets faced by the two investor types. Investment opportunity sets differ since the two types of agents have different productivities and face potentially heterogeneous sets of constraints. We contrast these SDFs to understand the effects of financial frictions onto the compensations for aggregate risks earned by our different agents.

We characterize the dynamics of the SDFs through their short-run dynamics, but we supplement these with horizon-dependent diagnostics as well. Short-run dynamics of an SDF S_t are summarized by its drift and diffusion (μ_S, σ_S), which pin down locally risk-free investment returns as well as local risk prices. But these short-run diagnostics miss potentially interesting intermediate- and long-horizon asset return properties. We thus augment the standard asset-pricing analysis with a computation of term structures of risk prices, which represent Sharpe ratios for dividend strips with various maturities. Our term structures are characterized using the recently-developed shock elasticities' tool kit.[2] They provide a useful joint summary of a model's state dynamics and asset prices at different horizons. Importantly, they give us another diagnostic tool to help distinguish various models.

At the same time, shock elasticities can be interpreted as nonlinear impulse response functions.[3] For instance, shock exposure elasticities for a given cashflow give us the sensitivity of such expected future cashflow to a normalized shock occuring today, and in the context of linear models with a Gaussian shock structure, they are identical to impulse response functions. Using those shock elasticities to parse the underlying economics of financial frictions' models can thus bring this literature somewhat closer to the traditional macroeconomic

[2]See Borovička et al. (2011), Hansen (2012), Hansen (2013). An accessible review treatment is provided in the handbook chapter Borovička and Hansen (2016).

[3]See Borovička, Hansen, and Scheinkman (2014).

DSGE literature. Indeed, the macro-finance literature with frictions has until now mostly focused on short-run dynamics and long-run averages. In a stationary equilibrium with state vector X_t, short-run dynamics are described by drift and diffusion coefficients (μ_X, σ_X), while long-run outcomes are characterized by the stationary distribution $p(x)$. Less often studied are medium-run transition distributions $p_t(x)$ and how they vary with time t. Because of their impulse response interpretation, shock elasticities provide a concise way to summarize a model's medium-run dynamics. For example, in nonlinear environments like ours, shock elasticities can be informative about the term structure of crisis and recovery probabilities, among other things.

We cast our model in continuous time, consistent with many recent papers in this literature. Continuous-time models with Brownian information are tractable due to their local normality and localized transition dynamics. These features often deliver quasi-analytical expressions for decision rules, allowing numerical procedures to avoid maximization steps. Such expressions remain available even when optimization problems involve financial constraints, as in our model. Even more important, localized transition dynamics imply a sparse transition matrix for the numerically discretized model, which greatly reduces computational costs. Finally, global solutions methods can be easier to implement in continuous time because occasionally-binding constraints and highly nonlinear dynamics in extreme parts of the state space boil down to an analysis of boundary conditions. Global solutions allow us to evaluate the importance of various models' nonlinearities and constraints.

We view part of our contribution as delivering a robust numerical solution method for a high-dimension non-linear stochastic general equilibrium model with occasionally-binding financial constraints. The equilibrium of our model reduces to a pair of coupled, nonlinear, second-order partial differential equations for agents' value functions, which have dimension equal to the number of state variables in the model. We solve these equations with an implicit finite difference scheme. In doing so, we take the standard approach of inserting the PDE nonlinearities into an iterative step, by augmenting the time-independent PDEs with a false time-derivative. At each iterative step, the discretized implicit scheme yields a large linear system to solve. The sparsity offered by continuous time delivers speed gains, but to speed things up even further, we leverage parallelization techniques and high-performance computing packages in C++.[4] Occasionally-binding constraints partition the state space into regions where constraints bind and where they are slack. Such partitions are endogenous hyper-surfaces in our state space, and solving for them numerically is a notoriously hard problem. We tackle this issue by employing another iterative procedure, embedded within

[4]For example, we have used Pardiso (https://www.pardiso-project.org/) for fast LU decompositions of our linear system.

the time-iterations for the PDEs. In particular, we iterate back and forth between agents' Euler inequalities and market clearing conditions until both equilibrium prices and these endogenous hyper-surfaces converge.

2 General Model

The model presented below is set in continuous time, $t \in [0, \infty)$. It builds on the model of Brunnermeier and Sannikov (2016b), adding heterogeneous recursive preferences, overlapping generations of agents, an exogenous TFP growth rate, both aggregate and idiosyncratic volatility shocks, and additional types of financial frictions.

Technology. There are two types of agents, experts and households, denoted by e and h, respectively. Each group has a continuum of agents indexed by j; the sets of experts and households are denoted by $\mathbb{J}_e$ and $\mathbb{J}_h$, respectively. Because of competition and homogeneity assumptions we introduce later, it suffices to consider a representative expert and a representative household.

Each agent produces output with a constant returns-to-scale technology taking only quality-adjusted capital as an input. In particular, an agent with $k_{j,t}$ units of capital produces $a_j k_{j,t}$ units of the unique consumption good. Within each group, agents' productivities a_j are homogeneous, and abusing notation somewhat, we write these productivities as a_h and a_e. We assume households are less productive than experts, $a_h \leq a_e$.

Quality-adjusted capital is accumulated via investment net of depreciation, as well as exogenous productivity gains. In addition, capital is subject to both aggregate and idiosyncratic capital-quality shocks (sometimes interpreted as TFP shocks or stochastic depreciation shocks). Mathematically, capital owned by agent j between t and $t + dt$ evolves as follows:

$$dk_{j,t} = k_{j,t} \left[(g_t + \iota_{j,t} - \delta) \, dt + \sqrt{s_t} \sigma \cdot dZ_t + \sqrt{\varsigma_t} dZ_{j,t} \right], \tag{1}$$

where g_t is exogenous productivity growth, $\iota_{j,t}$ denotes the investment rate, δ is depreciation, $\{Z_t\}_{t \geq 0}$ is a d-dimensional standard Brownian motion with independent components, and $\{Z_{j,t}\}_{t \geq 0}$ is a one-dimensional idiosyncratic Brownian motion, independent of Z and Z_{-j} (i.e., the idiosyncratic Brownian shocks hitting all other agents in the economy). Importantly, note that the law of motion for capital in (1) does not account for purchases and sales of capital, which also may occur.

Investment is subject to adjustment costs, which are paid out of current period output. By investing $\iota_{j,t}$, agent j pays $\Phi(\iota_{j,t}) k_{j,t}$, where $\Phi(\cdot)$ is an increasing and convex function satisfying $\Phi(0) = 0$, $\Phi'(0) = 1$. In applications, we set $\Phi(x) = \phi^{-1}[\exp(\phi x) - 1]$, which has the aforementioned properties.[5]

Exogenous States. The expected growth rate g_t, aggregate stochastic variance s_t, and idiosyncratic stochastic variance ς_t all evolve exogenously according to

$$dg_t = \lambda_g(\overline{g} - g_t)dt + \sqrt{s_t}\sigma_g \cdot dZ_t \tag{2}$$

$$ds_t = \lambda_s(\overline{s} - s_t)dt + \sqrt{s_t}\sigma_s \cdot dZ_t \tag{3}$$

$$d\varsigma_t = \lambda_\varsigma(\overline{\varsigma} - \varsigma_t)dt + \sqrt{\varsigma_t}\sigma_\varsigma \cdot dZ_t \tag{4}$$

Because Z_t is a $d \times 1$ vector, $\sigma_g, \sigma_s, \sigma_\varsigma$ are $d \times 1$ vectors. The processes (g, s, ς) are all mean-reverting processes, so we must keep track of them as state variables. In particular, s and ς are both Feller square root processes. Notice that g would be an Ornstein-Uhlenbeck process if s_t were constant, while s adds stochastic volatility. Together, this setup is reminiscent of long-run risk models (see for example Bansal and Yaron (2004)), with the inclusion of production and idiosyncratic shocks.

Financial Markets. Capital is freely traded and has (quality-adjusted) price q_t, which evolves as

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t} \cdot dZ_t]. \tag{5}$$

The coefficient μ_q and the $d \times 1$ vector σ_q are determined in equilibrium. Adjustment costs on investment create dynamics for q_t. Despite the presence of financial frictions, to be described shortly, financial markets are dynamically complete in this economy. Thus, define the unique stochastic discount factor (SDF)

$$dS_t = -S_t[r_t dt + \pi_t \cdot dZ_t]. \tag{6}$$

In (6), r is the short-term interest rate, and π denotes the $d \times 1$ vector of risk prices associated with each shock in Z. To ensure complete markets, we introduce zero-cost insurance contracts (futures) associated with the aggregate shocks Z_t, which have unit exposure and expected returns π. Idiosyncratic shocks are not traded, but they "wash out" in the aggregate analysis.

[5] This setup is equivalent to one in which production is made by $a_j A_{j,t}\tilde{k}_{j,t}$, where physical capital $\tilde{k}_{j,t}$ satisfies

$$d\tilde{k}_{j,t} = \tilde{k}_{j,t}(\iota_{j,t} - \delta)dt,$$

where the stochastic portion of agents' TFPs $A_{j,t}$ follows

$$dA_{j,t} = A_{j,t}\left(g_t dt + \sqrt{s_t}\sigma \cdot dZ_t + \sqrt{\varsigma_t}dZ_{j,t}\right),$$

and where adjustment costs are equal to $A_{j,t}\tilde{k}_{j,t}\Phi(\iota_{j,t})$. The efficiency units of capital are then $k_{j,t} := A_{j,t}\tilde{k}_{j,t}$.

Return-on-Capital. As a result of productivity differences, experts and households earn different cash flows, hence different returns, from holding capital. In particular, the return is defined by $dR^k_{j,t} := \frac{a_j k_{j,t} - \Phi(\iota_{j,t}) k_{j,t}}{q_t k_{j,t}} dt + \frac{d(q_t k_{j,t})}{q_t k_{j,t}}$, so by Itô's formula,

$$dR^k_{j,t} = \underbrace{\Big[\frac{a_j - \Phi(\iota_{j,t})}{q_t} + \iota_{j,t} - \delta + g_t + \mu_{q,t} + \sigma_{K,t} \cdot \sigma_{q,t}\Big]}_{:=\mu_{R,j,t}} dt + \underbrace{\Big[\sigma_{K,t} + \sigma_{q,t}\Big]}_{:=\sigma_{R,t}} \cdot dZ_t + \sqrt{\varsigma_t} dZ_{j,t}, \quad (7)$$

where $\sigma_{K,t} := \sqrt{s_t}\sigma$. Thus, expected returns for experts and households might differ due to different dividend yields $\frac{a_j - \Phi(\iota_{j,t})}{q_t}$, and also potentially due to different investment rates $\iota_{j,t}$.[6]

Overlapping Generations. To achieve a stationary wealth distribution in an economy with a variety of financial frictions, we assume a "perpetual youth" overlapping generations (OLG) structure, similar to Gârleanu and Panageas (2015). All agents perish independently at the Poisson rate λ_d. To keep the population size constant, newborn agents arrive at the same rate λ_d. Among newborns, a fraction ν are designated as experts, while $1 - \nu$ are households. Dying agents' wealth is pooled and redistributed equally to newborns, regardless of their occupation designation ("unintended bequests"). To ensure that these bequests are positive, we assume there are no markets to hedge these idiosyncratic death shocks, although adding partial insurance markets would not significantly alter the analysis.

Preferences. Experts and households have continuous-time recursive preferences of Duffie and Epstein (1992),

$$U_{j,t} = \mathbb{E}\Big[\int_0^\infty \varphi_j(c_{j,t+s}, U_{j,t+s}) ds \mid \mathcal{F}_t\Big], \quad (8)$$

where the utility aggregator φ_j is defined as

$$\varphi_j(c, U) := \rho_j \frac{1-\gamma_j}{1-\psi_j} U \Big(c^{1-\psi_j} [(1-\gamma_j)U]^{-\frac{1-\psi_j}{1-\gamma_j}} - 1\Big). \quad (9)$$

Within each group (households and experts), preferences are assumed identical. Hence, let $(\psi_e, \gamma_e, \rho_e)$ and $(\psi_h, \gamma_h, \rho_h)$ denote the preference parameters of experts and households, respectively. These parameters have the following interpretation: $1/\psi_j > 0$ denotes the elasticity of intertemporal substitution; $\gamma_j > 0$ denotes relative risk aversion; and $\rho_j > 0$ denotes the subjective discount rate.[7] Third, because of the Poisson death rate λ_d, both ρ_e and ρ_h should be interpreted as discounting inclusive of the death rate.

[6] In our set-up, since investment decisions are intra-temporal decisions, such decisions will follow a standard "q"-theory, and thus investment rates of the two types of agents will be identical in equilibrium.

[7] When $\psi_j = \gamma_j$, the preferences collapse to CRRA; when $\psi_j = \gamma_j = 1$, agents have logarithmic preferences.

Budgets, Constraints, and Optimization. In this subsection, we develop the optimization problems of a representative expert and household. To economize on notation, we subscript all individual-specific variables by their group label only, $j \in \{e, h\}$. This will ultimately be justified by the constellation of competition and homogeneity assumptions we introduce, by which it suffices to consider a representative expert and a representative household.

Agents manage capital and produce, subject to some financial frictions. To describe the frictions, it helps to first describe agents' balance sheets. For a quantity of capital $k_{j,t}$ an agent wants to purchase and hold between t and $t + dt$, they need to raise $q_t k_{j,t}$ in financing. They use their personal net worth $n_{j,t}$, equity issuances $(1 - \chi_{j,t})q_t k_{j,t}$, and risk-free short term debt $\chi_{j,t} q_t k_{j,t} - n_{j,t}$. They owe short term creditors $r_t dt$ per unit of short term debt issued, and pay out $(r_t + \sigma_{R,t} \cdot \pi_t)dt + \sigma_{R,t} \cdot dZ_t + \sqrt{\varsigma_t} dZ_{j,t}$ per unit of equity issued. Note that equity issuance is the only way for experts to reduce their exposure to idiosyncratic shocks, since such shocks are not traded. Equity investors do not receive any compensation for taking on idiosyncratic risk $Z_{j,t}$ since such risk is perfectly diversifiable. Finally, note that there is a natural incentive for experts (i.e. the most productive agents) to hold capital and sell equity, as opposed to simply holding a lower amount of capital: indeed, the experts earn a dividend yield – and thus an expected return on their capital – that is greater than households', and are effectively engaged into a "carry" trade, since they can afford to pay a risk price to households (on the equity they are issuing) that will sometimes be lower than what they earn from holding and operating this productive capital.

Although agents may issue equity to finance capital purchases, they must retain a fraction $\underline{\chi}_j \in [0, 1]$ of exposure to their assets. Hence, each agent faces the constraint

$$\chi_{j,t} \geq \underline{\chi}_j. \tag{11}$$

We allow $\underline{\chi}_e \neq \underline{\chi}_h$. This type of equity-issuance constraint, sometimes called a "skin-in-the-game" constraint, can be derived from a primitive moral hazard problem. Alternatively, such an equity constraint can be thought of as regulatory. Some papers, like He and Krishnamurthy (2012, 2013), allow partial equity-issuance (i.e., $0 < \underline{\chi}_j < 1$) and study how equilibrium dynamics are asymmetric around the points where constraints bind. Other papers, like Brunnermeier and Sannikov (2014), completely disallow equity-issuance (i.e., $\underline{\chi}_j = 1$). Appendix A.1 introduces additional leverage (sometimes referred to as Value-at-Risk) con-

In the case of the unitary elasticity of substitution ($\psi_j = 1$), the function φ takes the following form:

$$\varphi_j(c, U) := \rho_j(1 - \gamma_j)U\Big(\ln c - \frac{\ln[(1 - \gamma_j)U]}{1 - \gamma_j}\Big). \tag{10}$$

straints that will be analyzed in later sections of our paper.

Finally, agents may hedge their risk exposures through positions $\theta_{j,t}n_{j,t}$ in derivatives markets that pay $\pi_t dt + dZ_t$ per unit. This hedging is subject to the constraint

$$\theta_{j,t} \in \Theta_j. \tag{12}$$

We again allow $\Theta_e \neq \Theta_h$. In this setup, incomplete hedging partially intertwines experts' leverage and aggregate risk-taking decisions. Brunnermeier and Sannikov (2014) completely intertwines these decisions (i.e., $\Theta_e = \{0\}$), while Di Tella (2017) completely disentangles them (i.e., $\Theta_e = \Theta_h = \mathbb{R}^d$). Summarizing this discussion, figure 1 represents agents' balance sheets graphically.

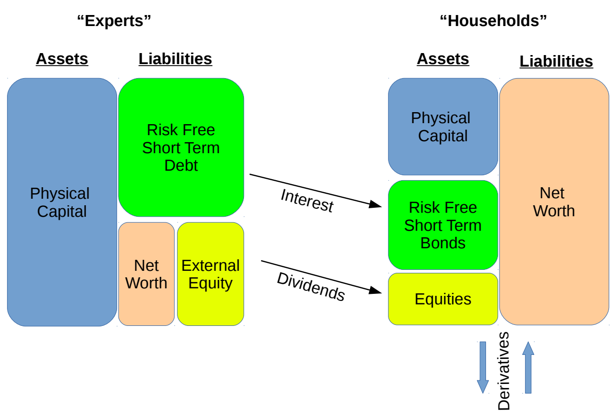

Figure 1: Balance sheets of experts and households.

The representative expert and household face the following dynamic budget constraints, for $j \in \{e, h\}$:

$$\frac{dn_{j,t}}{n_{j,t}} = \left(\mu_{n_j,t} - c_{j,t}/n_{j,t}\right) dt + \sigma_{n_j,t} \cdot dZ_t + \tilde{\sigma}_{n_j,t} dZ_{j,t} \tag{13}$$

where

$$\mu_{n_j,t} := r_t + (q_t k_{j,t}/n_{j,t})\,(\mu_{R,j,t} - r_t) - (1 - \chi_{j,t})\,(q_t k_{j,t}/n_{j,t})\sigma_{R,t} \cdot \pi_t + \theta_{j,t} \cdot \pi_t \tag{14}$$

$$\sigma_{n_j,t} := \chi_{j,t}(q_t k_{j,t}/n_{j,t})\sigma_{R,t} + \theta_{j,t} \tag{15}$$

$$\tilde{\sigma}_{n_j,t} := \chi_{j,t}(q_t k_{j,t}/n_{j,t})\sqrt{\varsigma_t}. \tag{16}$$

In agents' net worth evolutions, $(q_t k_{j,t}/n_{j,t})(\mu_{R,j,t} - r_t)$ is the excess return earned on the capital, and $(1-\chi_{j,t})(q_t k_{j,t}/n_{j,t})\sigma_{R,t}\cdot\pi_t$ is expected excess return compensation owed to outside equity investors. Once again, third-party equity investors are not compensated for the idiosyncratic risk embedded in the equity issued, since such risk is fully diversifiable. Agents choose $(\iota_j, k_j, \chi_j, \theta_j, c_j)$ to maximize utility (8) subject to their budget constraint (13), equity constraint (11), hedging constraint (12), short-sale constraint $k_{j,t}\geq 0$, and solvency constraint $n_{j,t}\geq 0$.

Competitive Equilibrium. A competitive equilibrium is a set of price and allocation processes, i.e., $(r_t,\pi_t,q_t)_{t\geq 0}$ and $(c_{j,t}, n_{j,t}, k_{j,t}, \chi_{j,t}, \theta_{j,t})_{j\in\mathbb{J}_e\cup\mathbb{J}_h, t\geq 0}$, such that agents solve their optimization problems, taking price processes as given, and the following market clearing conditions hold.

- *Goods market clearing:*

$$\int_{\mathbb{J}_e\cup\mathbb{J}_h} c_{j,t}dj = \int_{\mathbb{J}_e\cup\mathbb{J}_h}(a_j-\Phi(\iota_{j,t}))k_{j,t}dj. \tag{17}$$

- *Capital market clearing:*

$$\int_{\mathbb{J}_e\cup\mathbb{J}_h} k_{j,t}dj = K_t. \tag{18}$$

- *Equity market clearing:*

$$\int_{\mathbb{J}_e\cup\mathbb{J}_h}(1-\chi_{j,t})q_t k_{j,t}\sigma_{R,t}dj = \int_{\mathbb{J}_e\cup\mathbb{J}_h}\theta_{j,t}n_{j,t}dj. \tag{19}$$

- *Bond market clearing:*

$$\int_{\mathbb{J}_e\cup\mathbb{J}_h}(q_t k_{j,t}-n_{j,t})dj = 0. \tag{20}$$

We will look for a symmetric equilibrium of the model, in which all agents within the same class use the same strategy.

3 Equilibrium Characterization

Markov Equilibrium. Define experts' net worth share $w_t := \int_{\mathbb{J}_e} n_{j,t}dj/(q_t K_t)$.[8] The state variables in this economy are $(K_t, w_t, g_t, s_t, \varsigma_t)$. Because of the scaling property of the model, all growing quantities scale with K_t. Thus, in the de-trended economy $X_t := (w_t, g_t, s_t, \varsigma_t)$

[8]The wealth distribution is a state variable in our model. However, given the homogeneity properties of our model, it is sufficient to only keep track of the share of aggregate wealth in the hands of experts.

serves as a state variable. The state space is $\mathcal{X} := (0,1) \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$. Conjecture the following diffusive dynamics for X:

$$dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dZ_t,$$

where μ_X is a 4×1 vector, while σ_X is a $4 \times d$ matrix. Three of the components of X are specified exogenously, but the dynamics of w need to be determined in equilibrium.

Solution Method. In Appendix A.2, we apply a dynamic programming approach to solve agents' optimization problems, which delivers a pair of Hamilton-Jacobi-Bellman (HJB) equations. Each is a 4-dimensional second-order nonlinear partial differential equation (PDE) for agents' value functions. Next, in Appendix A.3, we use the market clearing conditions and constraints to solve for all equilibrium objects, in terms of the state variables and the value functions. By reinserting these equilibrium prices and dynamics in the HJB equations, the entire equilibrium fixed point problem boils down to solving a pair of PDEs for agents' value functions. As a baseline numerical method, we implement an implicit finite difference scheme, which augments the PDE with an artificial time-derivative ("false transient") in order to iterate on the nonlinearities in the PDE system. More details on this procedure, as well as comparisons with an explicit scheme, are contained in Appendix B.

In equilibrium, the presence of occasionally-binding constraints such as (11) manifest as endogenous partitions of the state space $\mathcal{X}$. These partitions are identified numerically using the complementary slackness conditions from agents' optimization problems. In general, at parts of the state space where the constraints are slack, equilibrium dictates a first-order nonlinear differential equation system for (q, χ_e, χ_h), which are the capital price and the endogenous inside equity shares. When any of the constraints bind, the corresponding part of this system becomes degenerate. Thus, the endogenous partitions are determined by solving a system of variational inequalities defined by the equilibrium. We solve these variational inequalities jointly with the value function PDEs. See Appendix B for more details.

Stochastic Discount Factors. The presence of financial frictions implies there are two SDFs in this economy, one for experts and one for households. In analyzing the model, we can examine the properties of shadow risk prices π_e and π_h corresponding to each of these SDFs. Since households are always marginal in outside equity markets, we have $\pi \equiv \pi_h$ always. See Appendix A.5 for details on the the derivations of these shadow risk prices.

Traditional and Non-Traditional Diagnostic Tools. Traditional model diagnostics in this literature involve local state variable dynamics (μ_X, σ_X), local SDF dynamics (r, π_e) and

(r, π_h), capital prices q and their dynamics (μ_q, σ_q), and the ergodic distribution of X_t.

We also explore several non-traditional diagnostics based on shock elasticities, which are methods to calculate term structures of risk exposures and risk prices. For example, the shock-exposure elasticity for some cash flow $\{G_t\}_{t \geq 0}$ answers the question: how risky is the horizon-t strip G_t? The corresponding shock-price elasticity answers: what risk price is associated with G_t? In our calculations in section 4, we analyze elasticities where $G = C$ is aggregate consumption in the economy, but also where $G = C_e, C_h$ denotes experts' or households' consumptions.

Shock elasticities are interesting objects because they incorporate horizon-dependence, unlike local SDF dynamics, as well as state-dependence, unlike unconditional asset price moments. Note also that the risk exposures and risk prices depend on what shock is under consideration. We have multiple sources of risk (i.e., TFP shocks, growth rate shocks, aggregate volatility shocks, idiosyncratic volatility shocks), and we examine shock elasticities to each of these shocks.

These shock elasticities can also be thought of as counterparts to impulse response functions, an equivalence that can be made precise in continuous-time Brownian environments. A shock occurring at time 0 results in some expected change in the cash flow G_t and its expected excess return, which are the exact quantities shock-exposure and shock-price elasticities deliver. This impulse response interpretation can bridge the gap between these financial frictions models and conventional macroeconomic analysis. See Appendix C for more details on shock elasticities.

4 Model Comparisons

Literature Nested. With this model setup, we are able to approximately nest several models in the recent literature on macroeconomics with financial frictions. See Table 1. The parameters $\bar{\beta}_e, \bar{\beta}_h$ are defined by the leverage constraint (21) in the appendix, to be considered in a later iteration of this paper.

Benchmark Model. We calibrate most parameters of the model to be broadly consistent with annual calibrations of other papers in the literature. The Brownian shocks Z_t have dimension four. We assume the shocks to exogenous processes (g, s, ς) are independent, and independent of the capital-quality shock to k. Thus, we set σ, σ_g, σ_s, and σ_ς to each have three zero entries and one non-zero entry. See Table 2 for benchmark values.

In our benchmark economic environment, households cannot produce any capital ($a_h = -\infty$), meaning that all the capital in the economy will always be held and operated by

Paper	Parameters	Notes
Basak and Cuoco (1998)	$a_h = -\infty$, $\underline{\chi}_e = 1$, $\bar{\beta}_e = \infty$, $\gamma_h = \psi_h = 1$, $\gamma_e = \psi_e$, $\sigma_g = \sigma_s = \sigma_\varsigma = 0$	We add production. We also add OLG for stationarity.
He and Krishnamurthy (2013)	$a_h = -\infty$, $\underline{\chi}_e < 1$, $\bar{\beta}_e = \infty$, $\gamma_h = \psi_h = 1$, $\gamma_e = \psi_e$, $\sigma_g = \sigma_s = \sigma_\varsigma = 0$	We add production. Their households also have labor income.
Brunnermeier and Sannikov (2014)	$a_h > -\infty$, $\underline{\chi}_e = \underline{\chi}_h = 1$, $\bar{\beta}_e = \infty$, $\Theta_e = \{0\}$, $\gamma_h = \psi_h = \gamma_e = \psi_e = 1$, $\sigma_g = \sigma_s = \sigma_\varsigma = 0$	Their main model is risk-neutral.
Adrian and Boyarchenko (2012)	$a_h > -\infty$, $\underline{\chi}_e = 1$, $\bar{\beta}_e < \infty$, $\Theta_e = \{0\}$, $\gamma_h = \psi_h = \gamma_e = \psi_e = 1$, $\sigma_g = \sigma_s = \sigma_\varsigma = 0$	Their households additionally face liquidity preference shocks.
Di Tella (2017)	$a_h = -\infty$, $\bar{\beta}_e = \infty$, $\Theta_e = \Theta_h = \mathbb{R}^d$, $\gamma_h = \gamma_e$, $\psi_h = \psi_e$, $\sigma_g = \sigma_s = 0$, $\sigma_\varsigma \neq 0$	They assume $\chi_{e,t} = \underline{\chi}_e$ always.
Gârleanu and Panageas (2015)	$a_h = a_e$, $\underline{\chi}_e = \underline{\chi}_h = 0$, $\bar{\beta}_e = \infty$, $\Theta_e = \Theta_h = \mathbb{R}^d$, $\gamma_e < \gamma_h$, $\sigma_g = \sigma_s = \sigma_\varsigma = 0$	Dynamically complete markets, but the wealth share of experts is still a state variable.
Bansal and Yaron (2004)	$a_h = a_e$, $\underline{\chi}_e = \underline{\chi}_h = 0$, $\bar{\beta}_e = \infty$, $\Theta_e = \Theta_h = \mathbb{R}^d$, $\gamma_e = \gamma_h$, $\sigma_g \neq 0$, $\sigma_s \neq 0$, $\sigma_\varsigma = 0$	Frictionless representative agent economy with stochastic growth and volatility.

Table 1: This table documents which parameter configurations from our model can approximate other models in the literature.

experts. Households and experts have identical preferences, with a risk aversion $\gamma_e = \gamma_h > 1$, and a unitary intertemporal elasticity of substitution ($\psi_h = \psi_e = 1$). Our experts in this benchmark environment face a partial skin-in-the-game constraint, $\underline{\chi}_e = 0.50$, meaning that experts must bear 50% of the aggregate risk related to the capital they operate. In our benchmark, we consider not only TFP shocks, but also growth rate shocks ($\|\sigma_g\| > 0$) and aggregate stochastic volatility shocks ($\|\sigma_s\| > 0$), but for the time being ignore idiosyncratic stochastic volatility shocks ($\|\sigma_\varsigma\| = 0$).

The resulting balance-sheet of agents is the following. Experts will finance their capital holdings via (a) their net worth, (b) the issuance of outside equity, and (c) the issuance of short term risk-free debt claims. Households will, in equilibrium, not hold any capital and will invest their entire net-worth in bonds and equity issued by experts.

In terms of constraints, this economic environment is closest to the intermediary asset pricing economy of He and Krishnamurthy (2013) (see Table 1), except that it includes a

Parameter	Value	Description
a_e	0.14	Experts' TFP
a_h	$-\infty$	Households' TFP
$\|\sigma\|$	0.04	Mean TFP vol.
ϕ	3	Adjustment cost
δ	0.05	Depreciation rate
$\bar{g}$	0	Mean exogenous TFP growth
λ_g	0.252	Mean-reversion exogenous TFP growth
$\|\sigma_g\|$	0.0141	Exogenous TFP growth vol.
$\bar{s}$	1	Mean "normalized" agg. TFP vol.
λ_s	0.156	Mean-reversion agg. TFP vol.
$\|\sigma_s\|$	0.132	Vol. "normalized" agg. TFP vol.
$\bar{\varsigma}$	0	Mean idio. TFP vol.
λ_ς	0	Mean-reversion idio. TFP vol.
$\|\sigma_\varsigma\|$	0	Vol. "normalized" idio. TFP vol
ρ_e	0.05	Expert rate of time preference
ρ_h	0.05	Household rate of time preference
ψ_e	1	Expert inverse EIS
ψ_h	1	Household inverse EIS
γ_e	3	Expert RRA
γ_h	3	Household RRA
λ_d	0.02	Birth/death rate
ν	0.1	Fraction of newborns designated experts
$\underline{\chi}_e$	0.50	Expert min skin-in-the-game
$\underline{\chi}_h$	1	Household min skin-in-the-game
$\bar{\beta}_e$	∞	Expert max VaR
$\bar{\beta}_h$	∞	Household max VaR
Θ_e	$\{0\}$	Expert hedging constraint set
Θ_h	$\mathbb{R}^4$	Household hedging constraint set

Table 2: Benchmark levels for parameters.

linear production technology, growth rate and aggregate variance shocks, and it generalizes the particular preference specification assumed in the original paper.

In this environment, due to the fact that both agents have unitary EIS and capital is always held entirely by experts, the price of capital q_t is constant,[9] so capital return volatility only stems from the volatility of the TFP shock. The relative wealth w_t is a mean reverting variable – when experts are relatively poor (i.e. w is low), they earn high expected returns and thus on average increase their net worth relative to households'.

Interaction of Financial Frictions with Other Shocks. Looking at the expert and household SDFs can be insightful about the financial frictions in the model. For example,

[9] This result can be derived from the consumption market clearing equation, and from the fact that unitary EIS agents keep their consumption-to-wealth ratios constant.

figure 2 shows expert and household risk prices to the TFP and aggregate volatility shocks, as a function of experts' wealth share (w) and stochastic variance (s).

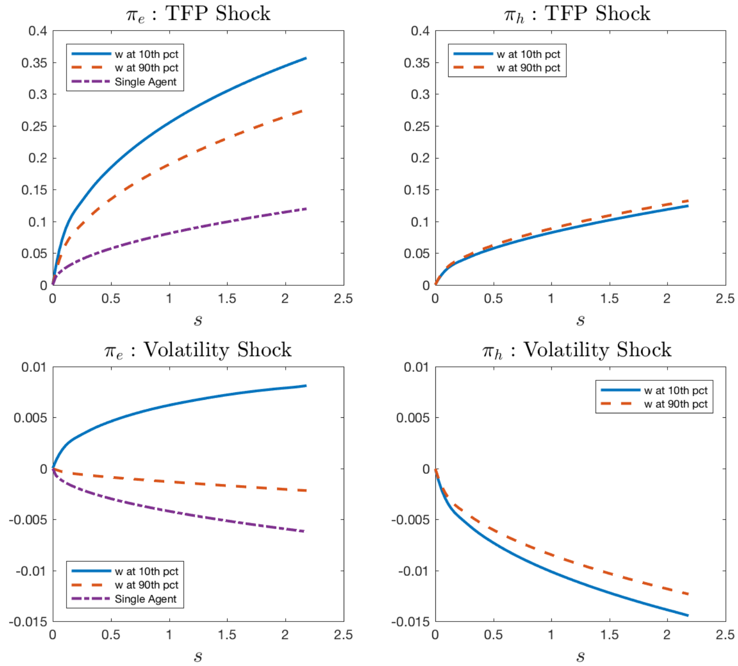

Figure 2: Local risk prices π_e and π_h for the TFP shock (first row) and volatility shock (second row). The "single-agent" risk price, from an economy with $\underline{\chi}_e = 0$, is plotted alongside π_e. Expected growth g is held fixed at $\overline{g}$.

As expected, experts' TFP risk price (top left panel) is falling in their relative wealth and rising in aggregate volatility. Households' TFP risk price (top right panel) is approximately the same as the risk price from a frictionless "single-agent" long-run risks model (e.g., Bansal and Yaron (2004)), which corresponds to $\underline{\chi}_e = 0$.[10]

Surprisingly, when their relative wealth is low enough, experts' volatility risk price (bottom left panel) switches from negative to positive, implying experts can be fond of volatility shocks. This occurs because the experts' expected returns on capital are strongly decreasing and convex in w, inheriting the shape of their TFP risk price (see equation (75) in Appendix A.5, which shows that TFP risk prices are proportional to w^{-1}). Jensen's inequality implies that a positive volatility shock benefits experts. This force outweighs the standard volatility aversion inherent in these preferences, which shows up in the negative household volatility risk price (bottom right panel) and the "single-agent" volatility risk price. Importantly, this insight is only available with the volatility shock introduced in the model: the TFP risk prices are increasing in volatility for both experts and households, which might suggest a

[10]In addition, we describe a single-agent long-run risks model, and its analytical solution for the unitary EIS case, in Appendix D.

distaste for volatility in the comparative static sense.

Occasionally-Binding Constraints. Given $\underline{\chi}_e < 1$, we might expect equity constraints to be occasionally-binding, as in He and Krishnamurthy (2013). However, it turns out that these constraints are always-binding unless experts and households differ in their preferences. See figure 3, which illustrates this point in a one-dimensional model where w is the only state variable (i.e., $\sigma_g = \sigma_s = 0$ so that g and s are non-stochastic). Thus, implicitly embedded in some of the "auxiliary assumptions" of He and Krishnamurthy (2013) and other models are assumptions about heterogeneous preferences.

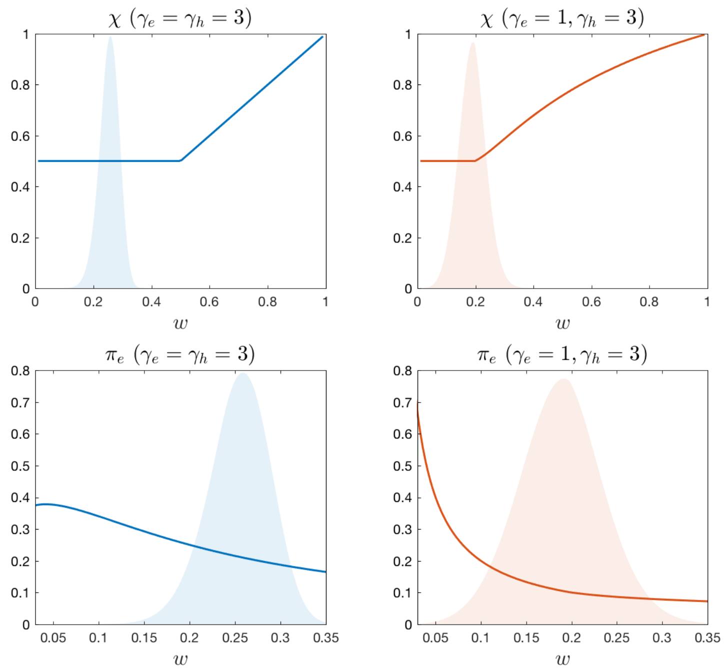

Figure 3: Expert skin-in-the-game χ (top row) and expert risk prices π_e (bottom row). Stationary densities are shaded in the background. We set $\sigma_g = \sigma_s = 0$. In the plots in the right column, we reduce γ_e from 3 to 1. All other parameters are given in Table 2.

With partial equity issuance ($\underline{\chi}_e < 1$), experts' financing constraint does not bind when w is sufficiently high. Under symmetric preferences, a non-binding constraint implies local Sharpe ratios earned by experts and households are equalized. In such an environment, the wealth share w is locally deterministic, moving purely due to births and deaths, thus exiting the unconstrained region in finite time. With homogeneous preferences, the ergodic density of w resides exclusively in the constrained region (top left panel).

On the other hand, with heterogeneous preferences, the ergodic density can include both the constrained and unconstrained region (top right panel). This results in much more nonlinearity in experts' risk prices than in the homogeneous preference case (bottom two panels). Such a situation can materialize because the unconstrained region remains stochastic (in the sense that $\sigma_w \neq 0$), as is standard in heterogeneous preference models.

Volatility Paradox. In Brunnermeier and Sannikov (2014), the authors emphasize a "volatility paradox" which states that lower fundamental volatility can generate an offsetting increase in non-fundamental price volatility, in the long run. This feature operates through an endogenous deterioration of experts' balance sheets: lower fundamental volatility reduces experts' return-on-capital, which drags their relative wealth to a lower long-run level. Here, we re-examine this mechanism when volatility is truly stochastic, rather than a comparative static.

First, we study the impulse response of w_t to an s-shock. We compute the impulse response using the concept of shock-exposure elasticities, as described above. To study this in a relatively clean way, we shut down the growth rate shocks ($\sigma_g = 0$) and specialize to the case of logarithmic utility ($\gamma_e = \gamma_h = 1$) so that volatility shocks are not priced. Figure 4 shows that a positive volatility shock generates an increase in w_t. The dynamic response of w_t is hump-shaped, due to the fact that w_t returns to its stationary distribution. There is no impact response (i.e., $w_{0+} = w_0$), due to the absence of any forces leading to a state-dependent capital price (e.g., non-unitary EIS or finite household productivity).

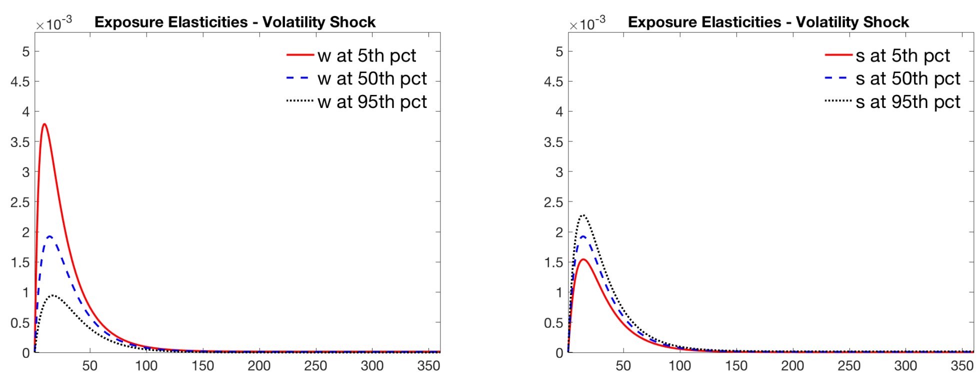

Figure 4: Shock-exposure elasticity of w_t to a volatility shock. In all models, experts and households have the same risk aversion, $\gamma_e = \gamma_h = \gamma$. Growth shocks are shut down, $\sigma_g = 0$. We set $\|\sigma_s\| = 0.132$, $\lambda_s = 0.156$, $\underline{\chi} = 1$, and $\gamma = 1$. All other parameters are given in Table 2.

An implication of this volatility IRF is that the equilibrium should feature a positive ergodic correlation between (w_t, s_t). But an interesting question is how strong this correlation is, and how it depends on the calibration of the model. To study this, we compute $\text{corr}(w_t, s_t)$ as a function of parameters $(\sigma_s, \lambda_s, \underline{\chi}, \gamma)$, shown in figure 5. First, notice that all the correlations are indeed positive.

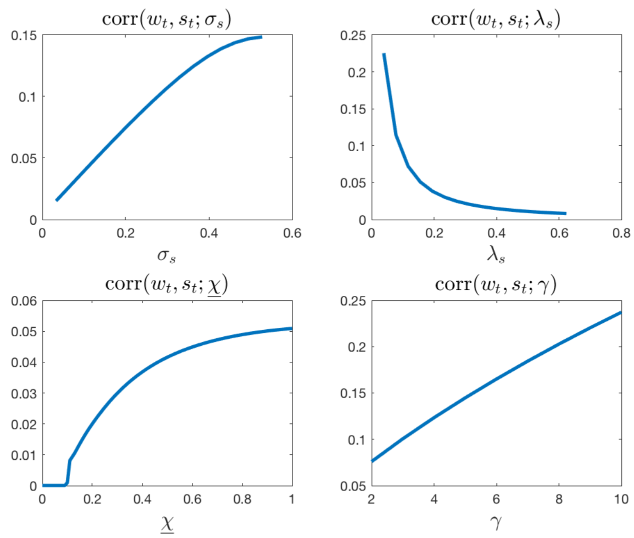

Figure 5: Ergodic correlations between (w_t, s_t) for different values of parameters $(\sigma_s, \lambda_s, \underline{\chi}, \gamma)$. In all models, experts and households have the same risk aversion, $\gamma_e = \gamma_h = \gamma$. Growth shocks are shut down, $\sigma_g = 0$. With the exception of the parameter being varied, we set $\|\sigma_s\| = 0.132$, $\lambda_s = 0.156$, $\underline{\chi} = 1$, and $\gamma = 1$. All other parameters are given in Table 2.

Focusing on the top row shows that, as volatility shocks are larger (higher σ_s) or more persistent (lower λ_s), the ergodic wealth-volatility correlation increases substantially. This helps explain strength of the comparative static results of Brunnermeier and Sannikov (2014). The bottom row shows that less equity-issuance (higher $\underline{\chi}$) or more risk-averse agents (higher γ) raises the (w_t, s_t) correlation. Intuitively, higher $\underline{\chi}$ or higher γ both scale experts' risk compensations, amplifying the effect of a volatility shock.

Are Experts More Productive or Risk-Tolerant? Among many alternatives, we focus on two competing hypotheses for why subsets of agents ("experts") take larger amounts of risk than others. Experts could be more productive managers of capital ($a_e > a_h$) or they could simply be more risk-tolerant ($\gamma_e < \gamma_h$). The two models closest to this comparison are Brunnermeier and Sannikov (2014) versus Gârleanu and Panageas (2015).

Figure 6 shows the endogenous capital distribution $\kappa := K_{e,t}/(K_{e,t} + K_{h,t})$ as a function of state variables (w, s), holding fixed $g = \overline{g}$. Both models feature regions where $\kappa \in (0, 1)$, but the heterogeneous productivity model showcases a large part of the state space where experts hold the entire capital stock. To understand this, consider the Merton portfolio $k_j^* = \frac{\mu_{R,j} - r}{\gamma_j \sigma_R}$ for agent j. With higher productivity, experts obtain a discretely larger expected return on capital than households ($\mu_{R,e} > \mu_{R,h}$) than households, so it is possible for households' desired portfolio to be negative ($k_h^* < 0$), leaving them on their no-shorting constraint. If experts and households faced the same returns ($\mu_{R,e} = \mu_{R,h}$) but their risk aversions differed ($\gamma_e < \gamma_h$), experts and households would both hold positive quantities of capital but at different scales ($k_e^* > k_h^* > 0$).

In a model with heterogeneous productivity, when households finally do start holding capital, capital prices fall sharply and risk compensations rise sharply, as seen in figure 7

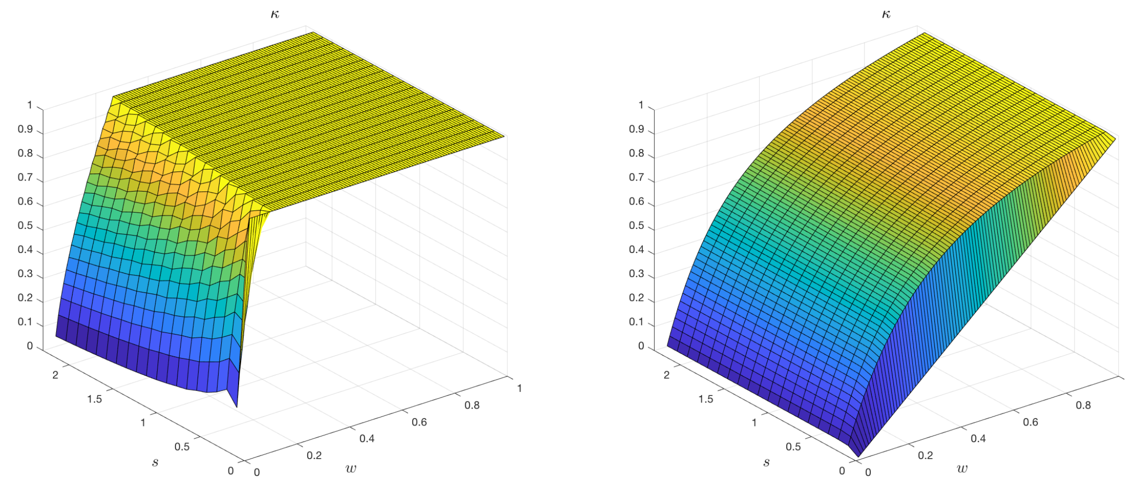

Figure 6: Expert capital share κ as a function of wealth share w and stochastic variance s. Expected growth g is held fixed at $\overline{g}$. Left panel has more productive experts ($a_h = 0.7 < a_e = 0.14$, $\gamma_e = \gamma_h = 3$, $\nu = 0.01, \lambda_d = 0.04$). Right panel has more risk-tolerant experts ($a_h = a_e = 0.14$, $\gamma_e = 2 < \gamma_h = 8$, $\nu = 0.10, \lambda_d = 0.02$). Parameters $(\gamma_e, \gamma_h, \nu, \lambda_d)$ are chosen such that the long-run mean of w_t is approximately the same in the two models. All other parameters are in Table 2.

(left panel). In contrast, a model with heterogeneous preferences features much smoother and more gradual risk price dynamics. Importantly, this comparison is made holding the wealth distribution relatively fixed across the two models (this is accomplished by varying parameters $(\gamma_e, \gamma_h, \nu, \lambda_d)$ to adjust the densities of w_t). In future analyses, we would like to experiment with imposing additional observational constraints in our model comparisons (e.g., similar average risk prices). Imposing observational constraints can impose strong discipline on the models in searching for distinguishing features.

These observed differences between experts' risk prices also show up in the shock-price elasticities of these models. The bottom row of figure 8 illustrates the term structure of experts' TFP risk prices. When experts' relative wealth is moderate (e.g., w_t at its median), the term structures between the models are similar. When experts are distressed (e.g., w_t at a low percentile), they look different: in the heterogeneous productivity model, the level of the term structure rises dramatically, and it becomes more negatively-sloped.

The top row shows that risk exposures in the heterogeneous productivity model are also higher when w_t is low, whereas they are invariant to w_t in the heterogeneous risk aversion model. To generate a closer link between the models on this dimension, we would need to calibrate non-unitary EIS in the heterogeneous risk aversion model, which we leave for a future iteration.

Another interesting difference between these models is embedded in experts' perception of

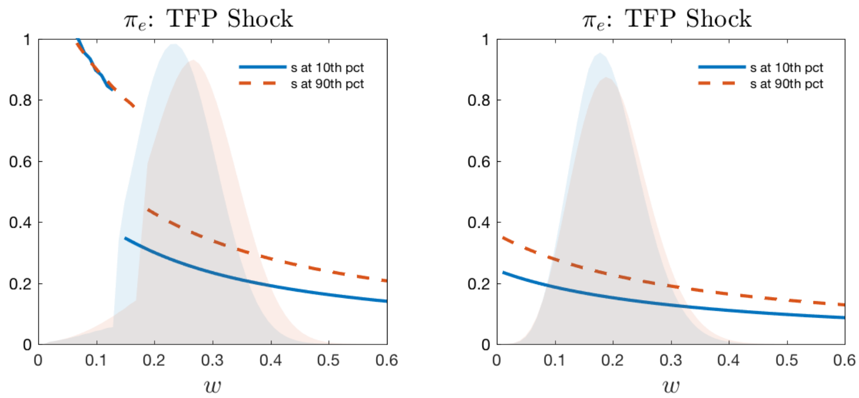

Figure 7: Experts' TFP risk prices π_e in models with heterogeneous productivity versus heterogeneous risk aversion. Stationary densities are shaded in the background. Expected growth g is held fixed at $\overline{g}$. Left panel has more productive experts ($a_h = 0.7 < a_e = 0.14$, $\gamma_e = \gamma_h = 3$, $\nu = 0.01, \lambda_d = 0.04$). Right panel has more risk-tolerant experts ($a_h = a_e = 0.14$, $\gamma_e = 2 < \gamma_h = 8$, $\nu = 0.10, \lambda_d = 0.02$). Parameters $(\gamma_e, \gamma_h, \nu, \lambda_d)$ are chosen such that the long-run mean of w_t is approximately the same in the two models. All other parameters are in Table 2.

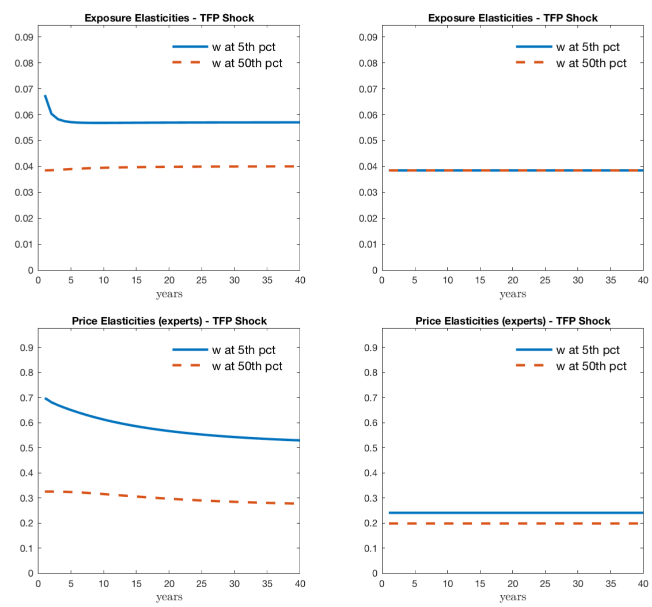

Figure 8: Shock-exposure and -price elasticities of aggregate consumption C_t to the TFP shock. Expected growth g is held fixed at $\overline{g}$. Left column has more productive experts ($a_h = 0.7 < a_e = 0.14$, $\gamma_e = \gamma_h = 3$, $\nu = 0.01, \lambda_d = 0.04$). Right column has more risk-tolerant experts ($a_h = a_e = 0.14$, $\gamma_e = 2 < \gamma_h = 8$, $\nu = 0.10, \lambda_d = 0.02$). Parameters $(\gamma_e, \gamma_h, \nu, \lambda_d)$ are chosen such that the long-run mean of w_t is approximately the same in the two models. All other parameters are in Table 2.

volatility shocks. Figure 9 shows shock-exposure and shock-price elasticities to a s_t-shock. In normal times (e.g., w_t at its median), the two models both produce similar volatility shock-exposure and shock-price elasticities. The positive sign and shape of the volatility shock-exposure elasticity are standard in long-run risk models: higher volatility of log consumption leads to higher future expected consumption through a Jensen effect. The positive sign of the volatility shock-price elasticity is due to the possibility of experts' demanding a positive volatility risk price, as in the discussion surrounding figure 2.

In bad times (e.g., w_t at its 5th percentile), the sign of both the exposure and price elasticities flip dramatically in the heterogeneous productivity model (left column). In such times, households are now holding capital, and a positive volatility shock further increases their desire to hold capital (this can also be seen in the left panel of figure 6). This leads a strong decline in capital prices, output, and consumption. Without these forces, experts liked volatility shocks, because their risk prices were strongly convex in w_t. With capital reallocation, π_e is less strongly convex in w_t, attenuating this force. More importantly, with an expected decline in aggregate consumption, there is a direct negative impact of a positive volatility shock.

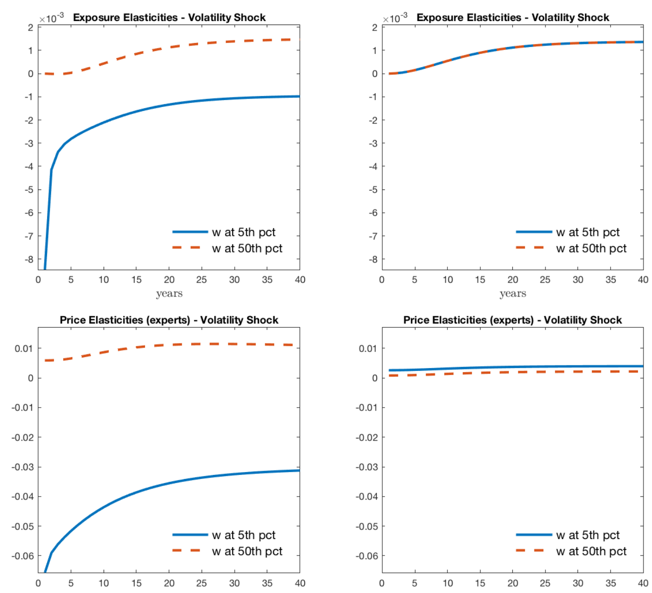

Figure 9: Shock-exposure and -price elasticities of aggregate consumption to the volatility shock. Expected growth g is held fixed at $\overline{g}$. Left column has more productive experts ($a_h = 0.7 < a_e = 0.14$, $\gamma_e = \gamma_h = 3$, $\nu = 0.01, \lambda_d = 0.04$). Right column has more risk-tolerant experts ($a_h = a_e = 0.14$, $\gamma_e = 2 < \gamma_h = 8$, $\nu = 0.10, \lambda_d = 0.02$). Parameters $(\gamma_e, \gamma_h, \nu, \lambda_d)$ are chosen such that the long-run mean of w_t is approximately the same in the two models. All other parameters are in Table 2.

Differential Consumption Dynamics. Because of financial frictions, the dynamics of experts' and households' consumptions can be very different. Figure 10 plots shock-exposure elasticities for C_e and C_h in the previous heterogeneous productivity model. Again, these can be thought of as nonlinear IRFs.

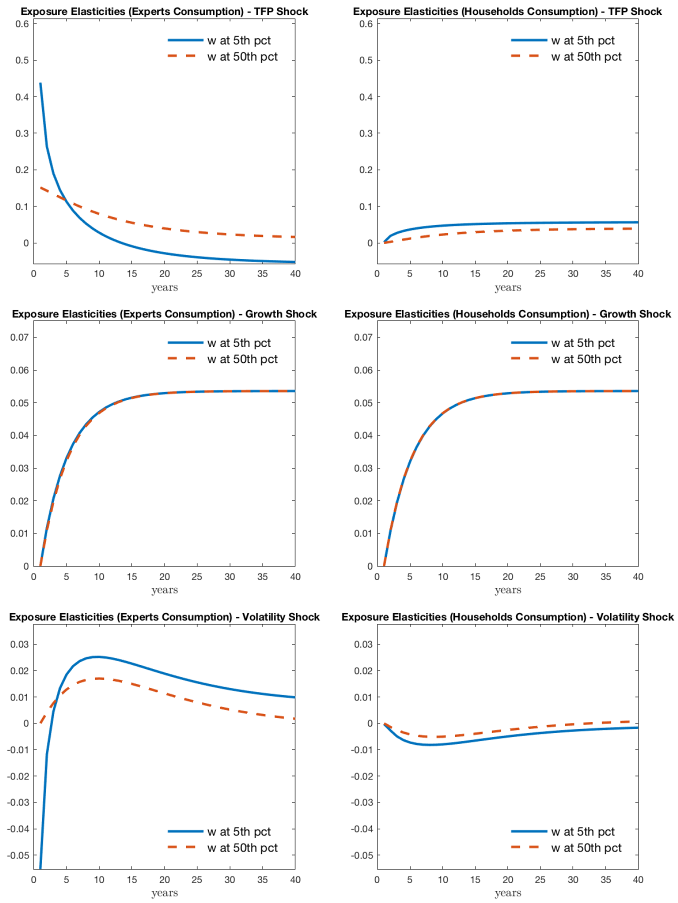

Figure 10: Shock-exposure elasticities for expert consumption C_e (left column) and household consumption C_h (right column). Expected growth g is held fixed at $\bar{g}$. The parameters for this model coincide with the heterogeneous productivity model ($a_h = 0.7 < a_e = 0.14$, $\gamma_e = \gamma_h = 3$, $\nu = 0.01, \lambda_d = 0.04$). All other parameters are in Table 2.

Because experts have leverage, their consumption responds much stronger to a positive TFP shock (top row). As w falls, expert leverage rises, so their consumption is accordingly more sensitive to a TFP shock. The growth rate shocks are shared perfectly in this economy, in the sense that C_e and C_h respond identically to a g-shock (middle row). This occurs because of the unitary EIS assumption of this model, something we plan to relax in the future. Finally, volatility shocks generate highly asymmetric responses between experts and households. In normal times (e.g., w at its median), one can see, by comparing the responses of C_e and C_h, that the response of $w := N_e/(N_e + N_h) \equiv C_e/(C_e + C_h)$ is qualitatively similar to the IRFs in figure 4. As before, higher volatility improves experts' investment opportunities relative to households, which improves their future consumption. In bad times (e.g., w at its 5th percentile), an increase in volatility lowers capital prices, which predominantly affects experts, so their consumption drops in the short run.

5 Conclusion

In this paper, we develop a general macroeconomic model with financial frictions that encompasses many recent papers in the literature. By varying parameters in the model, we compare implications of various financial frictions, preference constellations, and exogenous state variables. By introducing a model more general than those in the literature, we are also able to explore interactions between features of disparate models from the macro-finance literature. So far, we have made a subset of the comparisons we are interested in.

Readers can experiment with some of the parameter constellations we have solved by visiting our website (`https://modelcomparisons.shinyapps.io/modelcomparisonssite/`). There, we offer the ability to compare different model parameterizations directly, using some of the same diagnostic tools explored in this paper.

Going forward, we would make several additional comparisons with our model. How do equity constraints differ from leverage constraints? If agents can partially share risk by hedging in derivatives markets, in addition to issuing equity, how attenuated are the effects of other financial constraints? To what extent do asset price dynamics from a model with unhedgeable idiosyncratic uncertainty shocks resemble those from a model with other financial constraints? How do macro-prudential policies controlling equity and leverage constraints, among other things, affect asset prices, output growth, and welfare?

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A Equilibrium Derivation and Results

A.1 Portfolio Constraints

Adding leverage constraints within our framework is relatively simple. Assume agents' borrowing in the bond market $\chi_{j,t} q_t k_{j,t} - n_{j,t}$ is limited by the following,

$$\frac{\chi_{j,t} q_t k_{j,t}}{n_{j,t}} \sigma_{R,t} \leq \bar{\beta}_j, \tag{21}$$

where $\bar{\beta}_e \neq \bar{\beta}_h$ is allowed. This type of borrowing constraint, sometimes called a "value-at-risk" (VaR) constraint, can be derived from primitive agency frictions or attributed to regulatory requirements. Adrian and Boyarchenko (2012) incorporate a VaR constraint on their intermediaries. (In the current iteration, we have not explored $\bar{\beta}_j < \infty$, so we are effectively assuming no borrowing constraints.)

More generally, we can formulate our agents' optimization problem while encapsulating our specific constraints (skin-in-the-game, as well as leverage constraints) and clarify the role of constraints as risk-sharing impediments. Suppose agents choose $(c_j, k_j, \sigma_{n_j}, \tilde{\sigma}_{n_j})$ to maximize utility (8) subject to their budget constraint (13), shorting constraint $k_{j,t} \geq 0$, solvency constraint $n_{j,t} \geq 0$, and the time-varying following risk-bearing constraints:

$$\sigma_{n_j,t} \in \Sigma_{j,t} \quad \text{and} \quad \tilde{\sigma}_{n_j,t} \in \tilde{\Sigma}_{j,t}. \tag{22}$$

This formalism captures our constraints in the following way:

- Imposing the equity constraint (11) alone is equivalent to putting

$$\Sigma_{j,t} = \mathbb{R}^d \quad \text{and} \quad \tilde{\Sigma}_{j,t} = [\underline{\chi}_j \frac{q_t k_{j,t}}{n_{j,t}} \sqrt{\varsigma_t}, \infty).$$

- Imposing the leverage constraint (21) alone is equivalent to putting

$$\Sigma_{j,t} = \mathbb{R}^d \quad \text{and} \quad \tilde{\Sigma}_{j,t} = (-\infty, \bar{\beta}_j \frac{\sqrt{\varsigma_t}}{\sigma_{R,t}}].$$

- Imposing the hedging constraint (12) alone is equivalent to putting

$$\Sigma_{j,t} = \{y \sigma_{R,t} : y \geq 0\} + \Theta_j \quad \text{and} \quad \tilde{\Sigma}_{j,t} = \mathbb{R}_+.$$

Constraint combinations simply involve taking the intersection of the $\Sigma_{j,t}$ and $\tilde{\Sigma}_{j,t}$ sets above.

A.2 HJB equations

The continuation value U_j of any agent j is a function of its wealth, as well as the aggregate state vector X: $U_{j,t} = U_j(n_t, X_t)$. Using the analogue of dynamic programming results for recursive preferences, developed in Appendix A.4, we thus have the following Hamilton-Jacobi-Bellman (HJB) equation for any agent j (omitting the subscript $j \in \{e, h\}$ for notational simplicity):

$$
\begin{aligned}
0 = \max \varphi(c, U) + n\Big[\mu_n - c/n\Big]\partial_n U + \mu_X \cdot \partial_X U \\
+ \frac{1}{2} n^2 \left(\|\sigma_n\|^2 + \tilde{\sigma}_n^2\right) \partial_{nn} U + n\sigma_n \cdot \sigma_X' \partial_{nX} U + \frac{1}{2}\mathrm{tr}(\sigma_X' \partial_{XX'} U \sigma_X).
\end{aligned}
\tag{23}
$$

For households, the maximization (23) is over all possible choices of investment rate ι, capital-to-wealth holdings $\hat{k} \geq 0$, aggregate market hedges $\hat{\theta}$, and consumption-to-wealth ratio $\hat{c} \geq 0$. For experts, the maximization (23) is over all possible choices of investment rate ι, capital-to-wealth holdings $\hat{k} \geq 0$, consumption-to-wealth ratio $\hat{c} \geq 0$ and equity retention $\chi \geq \underline{\chi}$. Since φ_j in equation (9) is homogeneous of degree $1 - \gamma_j$ in (c, U), and since net worth evolutions in (13) are linear, the continuation utility U_j is also homogeneous of degree $1 - \gamma_j$ in net worth. In particular, agents' continuation value takes the form

$$
U_{j,t} = \frac{(\xi_j(X_t) n_{j,t})^{1-\gamma_j}}{1 - \gamma_j}, \quad j \in \{h, e\},
\tag{24}
$$

where ξ_j is some positive function. By using identities relating U and its partial derivatives derived in Appendix A.6, we can divide the HJB equation satisfied by experts' and household's value function by $(1 - \gamma_j)U_j$, and omitting the subscript $j \in \{e, h\}$ for notational simplicity, we obtain:

$$
\begin{aligned}
0 = \max \frac{\rho}{1-\psi}\left[(c/n)^{1-\psi}\xi^{\psi-1} - 1\right] + \mu_n - c/n + \mu_X \cdot \partial_X \ln \xi - \frac{\gamma}{2}\left(\|\sigma_n\|^2 + \tilde{\sigma}_n^2\right) \\
+ (1-\gamma)\sigma_n \cdot \sigma_X' \partial_X \ln \xi + \frac{1}{2}\left[\mathrm{tr}(\sigma_X' \partial_{XX'} \ln \xi \sigma_X) + (1-\gamma)\|\sigma_X' \partial_X \ln \xi\|^2\right]
\end{aligned}
\tag{25}
$$

For both households and experts, optimal investment only affects the ex-consumption drift μ_n of net worth, so it satisfies the first-order condition (dropping j subscripts)

$$
\Phi'(\iota) = q.
\tag{26}
$$

Define the optimal investment function $\iota(q) := (\Phi')^{-1}(q)$, as Φ' is invertible. When $\Phi(\iota) = \phi^{-1}[\exp(\phi\iota) - 1]$, we have the functional forms:

$$\iota(q) = \phi^{-1}\log(q) \tag{27}$$

$$\Phi(\iota(q)) = \phi^{-1}(q - 1). \tag{28}$$

Note that experts' and households' investment rate are equal, since their investment cost function is the same. Optimal consumption satisfies the "envelope condition"

$$\partial_c\varphi(c, U) = \partial_n U. \tag{29}$$

Using the form of the function φ, together with the homogeneity of the continuation value in (24), the envelope condition (29) becomes

$$c_{j,t} = \rho_j^{1/\psi_j}\xi_{j,t}^{1-1/\psi_j} n_{j,t}. \tag{30}$$

This equation also holds in the particular case $\psi_j = 1$, in which case the consumption-wealth ratio is constant equal to ρ_j. Next, substitute results (26) and (30) back into the HJB equation (23) to obtain for both experts and households (without j subscripts)

$$\begin{aligned} 0 = \max \frac{\psi}{1-\psi}\rho^{1/\psi}\xi^{1-1/\psi} - \frac{\rho}{1-\psi} + \mu_n - \frac{\gamma}{2}\Big(\big\|\sigma_n\big\|^2 + \tilde{\sigma}_n^2\Big) + \Big[\mu_X + (1-\gamma)\sigma_X\sigma_n\Big] \cdot \partial_X \ln\xi \\ + \frac{1}{2}\Big[\mathrm{tr}(\sigma_X'\partial_{XX'}\ln\xi\sigma_X) + (1-\gamma)\|\sigma_X'\partial_X\ln\xi\|^2\Big] \end{aligned}$$

Households maximize over all possible choices of leverage $qk_h/n_h \geq 0$ and market hedges θ, while experts maximize over all possible choices of leverage $qk_e/n_e \geq 0$ and equity retention $\chi \geq \underline{\chi}$. Maximizing over the remaining variables involves a basic portfolio choice. For households, this results in

$$\mu_{R,h} - r + (1-\gamma_h)(\sigma_X\sigma_R) \cdot \partial_X \ln\xi_h \leq \gamma_h\left(\sigma_{n_h} \cdot \sigma_R + \tilde{\sigma}_{n_h}\sqrt{\varsigma}\right)$$
$$\pi + (1-\gamma_h)\sigma_X'\partial_X\ln\xi_h = \gamma_h\sigma_{n_h}.$$

Combining these equations, we have households' Euler equation,

$$\begin{cases} \mu_{R,h} - r \leq \pi \cdot \sigma_R, & \text{if} \quad k_h = 0 \\ \mu_{R,h} - r = \pi \cdot \sigma_R + \gamma_h\varsigma qk_h/n_h, & \text{if} \quad k_h > 0. \end{cases} \tag{31}$$

In other words, when households' expected capital return is below what they can earn with exposure to aggregate risk via futures contracts, they do not hold any capital. When they do hold capital, the expected return on such capital is equal to compensation for aggregate risk (via $\pi \cdot \sigma_R$) plus a compensation for taking idiosyncratic risk (via $\gamma_h \varsigma q k_h / n_h$, which, as is discussed in Appendix A.4, can be viewed as the product of the risk exposure $\sqrt{\varsigma_t}$ multiplied by a shadow risk price $\gamma_h \sqrt{\varsigma_t}(1-\kappa_t)/(1-w_t)$, where κ_t will be the fraction of total capital held by experts). In the absence of idiosyncratic capital quality shocks (i.e. when $\varsigma = 0$), the Euler equation is modified to $\mu_{R,h} - r \leq \pi \cdot \sigma_R$, with equality when $k_h > 0$. Households' optimal risk allocations are given by

$$\sigma_{n_h} = \frac{q k_h}{n_h} \sigma_R + \frac{\theta_h}{n_h} = \frac{\pi}{\gamma_h} + \frac{1-\gamma_h}{\gamma_h} \sigma_X' \partial_X \ln \xi_h \tag{32}$$

$$\beta_h = \frac{q k_h}{n_h} \begin{cases} = \frac{\Delta_h^+}{\gamma_h \varsigma}, & \text{if } \varsigma > 0 \\ \geq 0, & \text{if } \varsigma = 0. \end{cases} \tag{33}$$

Equation (32) is the optimal household exposure to aggregate risks, which is a standard combination of mean-variance efficient portfolio and hedging demands. In the above, we have introduced:

$$\Delta_{h,t} := \mu_{R,h,t} - r_t - \pi_t \cdot \sigma_{R,t} \tag{34}$$

$\Delta_{h,t}$ is the gap between the households' expected return on capital and the risk-premium paid by the market. As discussed, when $\Delta_h < 0$, households decide not to hold any capital. When $\Delta_h > 0$, their leverage increase linearly with Δ_h. Finally, note that with "log" investors, the hedging demand disappears and the household's optimal wealth exposure to aggregate shocks σ_{n_h} is simply equal to the risk-price vector π. Experts' portfolio choice is similar. Their first-order conditions

$$\mu_{R,e} - r - (1-\chi)\pi \cdot \sigma_R + \chi(1-\gamma_e)(\sigma_X \sigma_R) \cdot \partial_X \ln \xi_e = \gamma_e \chi \left(\sigma_R \cdot \sigma_{n_e} + \sqrt{\varsigma} \tilde{\sigma}_{n_e}\right)$$
$$\pi \cdot \sigma_R + (1-\gamma_e)(\sigma_X \sigma_R) \cdot \partial_X \ln \xi_e \leq \gamma_e \left(\sigma_R \cdot \sigma_{n_e} + \sqrt{\varsigma} \tilde{\sigma}_{n_e}\right),$$

can be combined to yield an Euler equation:

$$\begin{cases} \pi \cdot \sigma_R \leq \mu_{R,e} - r, & \text{if} \quad \chi = \underline{\chi} \\ \pi \cdot \sigma_R = \mu_{R,e} - r, & \text{if} \quad \chi > \underline{\chi}. \end{cases} \tag{35}$$

The interpretation of equation (35) is straightforward: if the risk-premium $\pi_t \cdot \sigma_{R,t}$ required to be paid to the market for issuing equity is lower than the expected excess return that experts earn on their capital, they will issue as much equity as they can, and bounce against their skin-in-the-game constraint $\underline{\chi}$. Experts' optimal leverage is given by

$$\beta_e := \frac{\chi q k_e}{n_e} = \frac{1}{\gamma_e\left(\|\sigma_R\|^2 + \varsigma\right)}\Big[\Delta_e + \pi\cdot\sigma_R + (1-\gamma_e)(\sigma_X\sigma_R)\cdot\partial_X \ln\xi_e\Big], \tag{36}$$

and because they take all aggregate risks in equal proportions, $\sigma_{n_e} = \beta_e\sigma_R$. Note that the hedging motive is absent in the case where $\gamma_e = 1$. In the above, the "wedge" $\Delta_{e,t}$ is defined as follows:

$$\Delta_{e,t} := \underline{\chi}^{-1}[\mu_{R,e,t} - r_t - \pi_t\cdot\sigma_{R,t}] \tag{37}$$

$\Delta_{e,t}$ represents the incremental risk premium attained by experts, per unit of equity investment. These are "rents" accruing to experts, due to the presence of financial frictions. In particular, due to Euler equation (35), $\Delta_{e,t} > 0$ only when the equity-issuance friction is binding, i.e., $\chi_t = \underline{\chi}$.

To see that Δ_e represents an incremental private risk premium for experts, use the definition of Δ_e and the Euler equation (35) to obtain the following equation: $\mu_{R,e} - r = \chi(\pi\cdot\sigma_R + \Delta_e) + (1-\chi)\pi\cdot\sigma_R$. In particular, $\chi(\pi\cdot\sigma_R+\Delta_e)$ represents the experts' excess return to "inside equity" whereas $(1-\chi)\pi\cdot\sigma_R$ represents experts' payout to "outside equity" held by households. Simply by accounting, these sum to the excess return on assets, and Δ_e can thus be interpreted as the bonus return per unit of inside equity, of which there are χ units.

Finally, we substitute the optimal choices (32) and (36) back into the HJB equations. When doing this substitution, we use the fact that $\mu_{n_h} = r + \sigma_{n_h}\cdot\pi + \beta_h\Delta_h$ and that $\mu_{n_e} = r + \beta_e\left(\Delta_e + \sigma_R\cdot\pi\right)$. We also use the fact that $\tilde{\sigma}_{n_h} = \beta_h\sqrt{\varsigma}$ for households, and that $\sigma_{n_e} = \beta_e\sigma_R$ and that $\tilde{\sigma}_{n_e} = \beta_e\sqrt{\varsigma}$ for experts. For households, we obtain:

$$\begin{aligned}0 = &\frac{\psi_h}{1-\psi_h}\rho_h^{1/\psi_h}\xi_h^{1-1/\psi_h} - \frac{\rho_h}{1-\psi_h} + r + \frac{1}{2\gamma_h}\Big(\|\pi\|^2 + (\gamma_h\beta_h\sqrt{\varsigma})^2\Big) + \Big[\mu_X + \frac{1-\gamma_h}{\gamma_h}\sigma_X\pi\Big]\cdot\partial_X\ln\xi_h\\ &+\frac{1}{2}\Big[\mathrm{tr}(\sigma_X'\partial_{XX'}\ln\xi_h\sigma_X) + \frac{1-\gamma_h}{\gamma_h}\|\sigma_X'\partial_X\ln\xi_h\|^2\Big]\end{aligned} \tag{38}$$

Without idiosyncratic capital quality shocks (i.e. when $\varsigma = 0$), the term $\gamma_h\beta_h\sqrt{\varsigma}$ disappears and we obtain the complete market, recursive preference HJB equation. For experts:

$$\begin{aligned}0 = &\frac{\psi_e}{1-\psi_e}\rho_e^{1/\psi_e}\xi_e^{1-1/\psi_e} - \frac{\rho_e}{1-\psi_e} + r + \frac{1}{2\gamma_e}\frac{(\Delta_e+\pi\cdot\sigma_R)^2}{\|\sigma_R\|^2+\varsigma} + \Big[\mu_X + \frac{1-\gamma_e}{\gamma_e}\Big(\frac{\Delta_e+\pi\cdot\sigma_R}{\|\sigma_R\|^2+\varsigma}\Big)\sigma_X\sigma_R\Big]\cdot\partial_X\ln\xi_e\\ &+\frac{1}{2}\Big[\mathrm{tr}(\sigma_X'\partial_{XX'}\ln\xi_e\sigma_X) + \frac{1-\gamma_e}{\gamma_e}(\sigma_X'\partial_X\ln\xi_e)'\Big[\gamma_e\mathbb{I}_d + (1-\gamma_e)\frac{\sigma_R\sigma_R'}{\|\sigma_R\|^2+\varsigma}\Big]\sigma_X'\partial_X\ln\xi_e\Big]\end{aligned} \tag{39}$$

In the above, $\mathbb{I}_d$ is the $d\times d$ identity matrix. Aside from the idiosyncratic risk that households are forced to bear, their HJB equation is a standard equation for a complete-markets investor with recursive preferences. Experts HJB equation additionally has effects of financial frictions, including the presence of the bonus risk premium Δ_e and the inability to hedge aggregate risks associated with (g, s, ς). Boundary conditions for (38) and (39) will be discussed below. Note also that the same PDEs hold for unitary IES investors, except that the term $\frac{\psi}{1-\psi}\rho^{1/\psi}\xi^{1-1/\psi} - \frac{\rho}{1-\psi}$ is replaced by $\rho\left(\ln\rho - \ln\xi\right) - \rho$.[11]

A.3 Solving for equilibrium dynamics

To characterize equilibrium, we must determine (i) q and its dynamics as a function of X; (ii) r, π, Δ_e and Δ_h as a function of X; and (iii) the dynamics of X. Because of the constraints $\chi \geq \underline{\chi}$ and $k_h \geq 0$, the state space must be partitioned into regions in which various constellations of constraints bind. To do this, define $\kappa_t := \frac{K_{e,t}}{K_t}$ to be the fraction of capital managed by experts. There are 3 cases to consider, by considering the Euler inequalities (31) and (35) for households and experts:[12]

- $\kappa = 1$ and $\chi > \underline{\chi}$ – In this case, all the capital is managed by experts, who are wealthy enough that their skin-in-the-game constraint is not binding. In this region, we have:

$$\mu_{R,e} - r = \pi \cdot \sigma_R > \mu_{R,h} - r$$

- $\kappa = 1$ and $\chi = \underline{\chi}$ – In this case, all the capital is managed by experts, but their skin-in-the-game constraint is now binding. In this region, we have:

$$\mu_{R,e} - r > \pi \cdot \sigma_R > \mu_{R,h} - r$$

- $\kappa < 1$ and $\chi = \underline{\chi}$ – In this case, experts' net worth, compared to the aggregate wealth in the economy, is too small, experts' risk-bearing capacity is low and some of the capital has to be held by household, whose productivity is lower than the experts'. In this region, we have:

$$\mu_{R,e} - r > \mu_{R,h} - r \geq \pi \cdot \sigma_R$$

[11] Note that this formula can easily be obtained using L'Hôpital's rule, noticing that:

$$\lim_{\psi\to 1} \frac{\psi\rho^{1/\psi}\xi^{1-1/\psi}}{1-\psi} - \frac{\rho}{1-\psi} = \rho\left(\ln\rho - \ln\xi\right) - \rho$$

[12] This applies when $a_e > a_h$ holds. If so, the case $\chi > \underline{\chi}$ and $\kappa < 1$ can be ruled out. Indeed, in such a case, agents' Euler equations (31) and (35) imply that $\mu_{R,e} - r = \mu_{R,h} - r$, which contradicts (7). We need to consider this case when $a_e = a_h$, which is treated in subsection A.3.1.

In the absence of idiosyncratic capital quality shocks (i.e. when $\varsigma = 0$), the last inequality is actually an equality – i.e. $\mu_{R,h} - r = \pi \cdot \sigma_R$. In other words, since households have no constraints, their expected return on capital must be equal to the market compensation for aggregate risk.

Combining these conditions with the definitions of Δ_e and Δ_h in (37) and (34), we completely summarize the constraints with the following complementary slackness conditions:

$$0 = \min(1 - \kappa, \Delta_h^+ - \Delta_h) \tag{40}$$

$$0 = \min(\chi - \underline{\chi}, \Delta_e) \tag{41}$$

$$0 = (1 - \kappa)(\chi - \underline{\chi})(a_e - a_h). \tag{42}$$

We will use these conditions to determine where constraints bind.

Before considering the state space partitions coming from constraints, we make use of some equilibrium conditions that apply across the state space. First, the goods market clearing condition (17) implies

$$q(1 - w)\rho_h^{1/\psi_h}\xi_h^{1-1/\psi_h} + qw\rho_e^{1/\psi_e}\xi_e^{1-1/\psi_e} = (1 - \kappa)a_h + \kappa a_e - \Phi(\iota(q)). \tag{43}$$

Equation (43) relates q and κ to the state variables, conditional on ξ_h and ξ_e. Notice that the left-hand-side (as a function of q) is strictly increasing, while the right-hand-side is strictly decreasing, yielding a unique q that satisfies the goods market clearing condition above. Notice also that in the unitary IES case, when all the capital in the economy is held by experts (i.e. $\kappa = 1$), the price of capital is constant, simply equal to the ratio of (a) the dividend yield $a_e - \Phi(\iota(q))$ divided by (b) a weighted average rate of time preference $w\rho_e + (1 - w)\rho_h$. With $\Phi(x) = \phi^{-1}[\exp(\phi x) - 1]$, we use (28) in equation (43) to get the following special case:

$$q = \frac{(1 - \kappa)a_h + \kappa a_e + 1/\phi}{(1 - w)\rho_h^{1/\psi_h}\xi_h^{1-1/\psi_h} + w\rho_e^{1/\psi_e}\xi_e^{1-1/\psi_e} + 1/\phi} \tag{44}$$

Next, the dynamics of aggregate capital are derived from time-differentiating the capital market clearing condition (18), using the laws of motion for individual capital stocks in (1), the common investment rate defined by (26), and the law of large numbers:

$$\frac{dK_t}{K_t} = \mu_{K,t}dt + \sigma_{K,t} \cdot dZ_t \tag{45}$$

$$\mu_{K,t} := g_t + \iota(q_t) - \delta. \tag{46}$$

To determine the dynamics of the wealth share w_t, combine agents' net worth dynamics with their portfolio choices (i.e., combine (13) with (32) for households and with (36) for experts), and the law of large numbers, to obtain evolutions for aggregate household and expert net worths $N_{h,t} := \int_{\mathbb{J}_h} n_{j,t}dj$ and $N_{e,t} := \int_{\mathbb{J}_e} n_{j,t}dj$:

$$\frac{dN_{h,t}}{N_{h,t}} = \Big[r_t - \rho_h^{1/\psi_h}\xi_h^{1-1/\psi_h} - \lambda_d + \sigma_{n_h,t}\cdot\pi_t + \beta_{h,t}\Delta_{h,t} + \frac{(1-\nu)\lambda_d}{1-w_t}\Big]dt + \sigma_{n_h,t}\cdot dZ_t \tag{47}$$

$$\frac{dN_{e,t}}{N_{e,t}} = \Big[r_t - \rho_e^{1/\psi_e}\xi_e^{1-1/\psi_e} - \lambda_d + \sigma_{n_e,t}\cdot\pi_t + \beta_{e,t}\Delta_{e,t} + \frac{\nu\lambda_d}{w_t}\Big]dt + \sigma_{n_e,t}\cdot dZ_t. \tag{48}$$

The terms containing λ_d represent contributions from OLG. Using risk choices (32) and (36), combined with equity market clearing (19) and the definitions of w and κ, we have

$$\sigma_{n_h} = \frac{1-\chi\kappa}{1-w}\sigma_R \tag{49}$$

$$\sigma_{n_e} = \frac{\chi\kappa}{w}\sigma_R. \tag{50}$$

By Itô's formula, the wealth share $w = \frac{N_e}{N_e+N_h}$ evolves as

$$dw = w(1-w)\Big[\frac{dN_e}{N_e} - \frac{dN_h}{N_h}\Big] - w(1-w)\Big[w\frac{d[N_e]}{N_e^2} - (1-w)\frac{d[N_h]}{N_h^2} + (1-2w)\frac{d[N_e,N_h]}{N_eN_h}\Big].$$

Using (47)-(48) and (49)-(50), and making several simplifications, the result is

$$\begin{aligned}\mu_w &= w(1-w)\Big[\rho_h^{1/\psi_h}\xi_h^{1-1/\psi_h} - \rho_e^{1/\psi_e}\xi_e^{1-1/\psi_e} + \beta_e\Delta_e - \beta_h\Delta_h\Big] \\ &\quad + (\chi\kappa - w)\sigma_R\cdot(\pi - \sigma_R) + \lambda_d(\nu - w)\end{aligned} \tag{51}$$

$$\sigma_w = (\chi\kappa - w)\sigma_R. \tag{52}$$

Together with the exogenous dynamics in (2), (3), and (4), the endogenous dynamics in (51)-(52) fully describe the dynamics of X_t, i.e.,

$$\mu_X = \Big(\mu_w,\ \lambda_g(\bar{g}-g),\ \lambda_s(\bar{s}-s),\ \lambda_\varsigma(\bar{\varsigma}-\varsigma)\Big)' \tag{53}$$

$$\sigma_X = \Big(\sigma_w,\ \sqrt{s}\sigma_g,\ \sqrt{s}\sigma_s,\ \sqrt{\varsigma}\sigma_\varsigma\Big)'. \tag{54}$$

By Itô's formula, the dynamics of q_t are

$$\begin{aligned}dq(X_t) &= \Big[\mu_X(X_t)\cdot\partial_X q(X_t) + \frac{1}{2}\mathrm{tr}(\sigma_X(X_t)'\partial_{XX'}q(X_t)\sigma_X(X_t))\Big]dt \\ &\quad + \Big[\sigma_X(X_t)'\partial_X q(X_t)\Big]\cdot dZ_t.\end{aligned} \tag{55}$$

Since μ_X does not depend directly on μ_q, the drift term may be obtained simply from the Itô's formula expansion:

$$\mu_q = \frac{1}{q}\Big[\mu_X \cdot \partial_X q + \frac{1}{2}\text{tr}(\sigma_X' \partial_{XX'} q \sigma_X)\Big]. \tag{56}$$

On the other hand, σ_X depends on σ_q, constituting a two-way feedback loop. We can solve this loop by substituting (54) into σ_q in (55), using $\sigma_R = \sigma_K + \sigma_q$:

$$\sigma_q = \frac{(\chi\kappa - w)\,(\partial_w \ln q)\,\sqrt{s}\sigma + (\partial_g \ln q)\,\sqrt{s}\sigma_g + (\partial_s \ln q)\,\sqrt{s}\sigma_s + (\partial_\varsigma \ln q)\,\sqrt{\varsigma}\sigma_\varsigma}{1 - (\chi\kappa - w)\partial_w \ln q} \tag{57}$$

This is a $d \times 1$ equation. Conditional on knowing χ and κ, if we know the price function q across the state space, we know the capital price volatility vector σ_q, as well as the wealth share volatility vector σ_w. Note that this generates capital return volatility equal to

$$\sigma_R = \frac{\sqrt{s}\sigma + (\partial_g \ln q)\,\sqrt{s}\sigma_g + (\partial_s \ln q)\,\sqrt{s}\sigma_s + (\partial_\varsigma \ln q)\,\sqrt{\varsigma}\sigma_\varsigma}{1 - (\chi\kappa - w)\partial_w \ln q}. \tag{58}$$

Finally, we solve for the stochastic discount factor coefficients (r, π). Time-differentiate the bond market clearing condition (20), using the evolutions of N_h and N_e in (47) and (48), and K in (45). By equating the drift terms:

$$r + (1-w)\Big(\sigma_{n_h} \cdot \pi + \beta_h \Delta_h - \rho_h^{1/\psi_h} \xi_h^{1-1/\psi_h}\Big) + w\Big(\sigma_{n_e} \cdot \pi + \beta_e \Delta_e - \rho_e^{1/\psi_e} \xi_e^{1-1/\psi_e}\Big) \\ = \mu_q + \mu_K + \sigma_K \cdot \sigma_q \tag{59}$$

By equating the diffusion terms:

$$(1-w)\sigma_{n_h} + w\sigma_{n_e} = \sigma_R. \tag{60}$$

To solve for r, substitute (60) into (59) and rearrange:

$$r = \mu_q + \mu_K + \sigma_K \cdot \sigma_q - \sigma_R \cdot \pi - (1-w)\Big(\beta_h \Delta_h - \rho_h^{1/\psi_h} \xi_h^{1-1/\psi_h}\Big) \\ - w\Big(\beta_e \Delta_e - \rho_e^{1/\psi_e} \xi_e^{1-1/\psi_e}\Big). \tag{61}$$

To solve for π, substitute exposures σ_{n_h} from (32) and σ_{n_e} from (50) into (60) to obtain:

$$\pi = \gamma_h \frac{1 - \chi\kappa}{1 - w}\sigma_R + (\gamma_h - 1)\sigma_X' \partial_X \ln \xi_h. \tag{62}$$

Therefore, we have solved for $(q, r, \pi, \mu_q, \sigma_q, \mu_X, \sigma_X)$, taking as given χ, κ, Δ_e, Δ_h, ξ_h, ξ_e, and all derivatives of ξ_h and ξ_e. In the next step, we solve for χ, κ, Δ_e, and Δ_h from the following conditions. First, combining experts' risk choice (36) with households' risk price (62), we get an equation relating χ, κ, and Δ_e:

$$\Delta_e = \gamma_e \frac{\chi\kappa}{w}(\|\sigma_R\|^2 + \varsigma) - \gamma_h \frac{1-\chi\kappa}{1-w}\|\sigma_R\|^2 - \sigma_R' \sigma_X' \partial_X \ln\left(\frac{\xi_h^{\gamma_h - 1}}{\xi_e^{\gamma_e - 1}}\right). \tag{63}$$

Second, by taking the difference $\mu_{R,e} - \mu_{R,h} = \mu_{R,e} - r - \pi \cdot \sigma_R + \pi \cdot \sigma_R - (\mu_{R,h} - r)$, using the definitions of $\mu_{R,e}$ and $\mu_{R,h}$ in (7), along with the definitions of Δ_e and Δ_h, we obtain

$$\Delta_h = \underline{\chi}\Delta_e - \frac{a_e - a_h}{q}. \tag{64}$$

Equations (63) and (64) completely solve for (Δ_e, Δ_h) given (κ, χ) and the other equilibrium objects. It remains to solve for (κ, χ).

To do this, recall the equation for households' capital holdings (33), which says

$$\Delta_h^+ = \frac{1-\kappa}{1-w}\gamma_h \varsigma. \tag{65}$$

Combine (65) with (63)-(64) to solve for $\Delta_h^+ - \Delta_h$. Substituting the result into the slackness condition (40) yields

$$0 = \min\Big\{1-\kappa,\, w\gamma_h(1-\chi\kappa)\|\sigma_R\|^2 + w\gamma_h\Big(\frac{1-\kappa}{\underline{\chi}}\Big)\varsigma - (1-w)\gamma_e\chi\kappa(\|\sigma_R\|^2 + \varsigma) + w(1-w)\frac{a_e - a_h}{\underline{\chi} q} + w(1-w)\sigma_R'\sigma_X'\partial_X \ln\left(\frac{\xi_h^{\gamma_h-1}}{\xi_e^{\gamma_e-1}}\right)\Big\}. \tag{66}$$

We may substitute $\chi = \underline{\chi}$ everywhere in this equation, due to (42). Given (ξ_e, ξ_h), equation (66) is actually a standalone variational inequality (differential equation wrapped inside of a min operator) for κ, since q can be expressed solely as a function of κ through (44), and since both σ_X and σ_R can be expressed solely in terms of χ, κ, q, and $\partial_X q$ through (54) and (58). By inspection, the boundary condition $\kappa(0, g, s, \varsigma) = 0$ will be satisfied automatically.

On the other hand, substituting Δ_e from (63) into the slackness condition (41) yields

$$0 = \min\Big\{\chi - \underline{\chi},\, (1-w)\gamma_e\chi\kappa(\|\sigma_R\|^2 + \varsigma) - w\gamma_h(1-\chi\kappa)\|\sigma_R\|^2 - w(1-w)\sigma_R'\sigma_X'\partial_X \ln\left(\frac{\xi_h^{\gamma_h-1}}{\xi_e^{\gamma_e-1}}\right)\Big\}. \tag{67}$$

We may substitute $\kappa = 1$ everywhere in this equation, due to (42). Given ξ_e, ξ_h, equation

(67) is actually an algebraic equation for χ. To solve, substitute σ_X and σ_R into the second term in the minimum, obtaining

$$0 = \min\Big\{\chi - \underline{\chi},\ (1-w)\gamma_e\varsigma(\partial_w \ln q)^2(\chi - w)^3 + (1-w)\gamma_e\varsigma(\partial_w \ln q)(w\partial_w \ln q - 2)(\chi - w)^2$$
$$+ \Big[((1-w)\gamma_e + w\gamma_h)\|D_{\tilde{X}}\|^2 + (1-w)\gamma_e\varsigma(1 - 2w\partial_w \ln q) + (\partial_w \ln q)D_{\xi,\tilde{X}} - D_{\xi,w}\Big](\chi - w)$$
$$+ w(1-w)(\gamma_e - \gamma_h)\|D_{\tilde{X}}\|^2 + w(1-w)\gamma_e\varsigma - D_{\xi,\tilde{X}}\Big\}. \tag{68}$$

In (68), $\tilde{X} := (g, s, \varsigma)$ refers to the exogenous state variables in the state vector X and we have defined[13]

$$\sigma_{\tilde{X}} := \left(\sqrt{s}\sigma_g, \sqrt{s}\sigma_s, \sqrt{\varsigma}\sigma_\varsigma\right)' \tag{69}$$
$$D_{\tilde{X}} := \sigma_K + \sigma_{\tilde{X}}'\partial_{\tilde{X}} \ln q \tag{70}$$
$$D_{\xi,w} := \|D_{\tilde{X}}\|^2\partial_w\left[\ln(\xi_h^{\gamma_h-1}) - \ln(\xi_e^{\gamma_e-1})\right] \tag{71}$$
$$D_{\xi,\tilde{X}} := (\sigma_{\tilde{X}}D_{\tilde{X}})\cdot\partial_{\tilde{X}}\left[\ln(\xi_h^{\gamma_h-1}) - \ln(\xi_e^{\gamma_e-1})\right]. \tag{72}$$

When, $\chi > \underline{\chi}$, equation (68) is a cubic equation for $\chi - w$. The analysis is simpler in two special cases. First, when $\varsigma = 0$, (68) becomes a linear equation which thus has a unique solution. Second, in the absence of exogenous state variables $\tilde{X}$, one can show that $\chi = \max(w, \underline{\chi})$ if and only if $\gamma_e = \gamma_h$.

Thus, equations (66)-(67) are enough to solve for (κ, χ) across the state space. Substituting these back into (63)-(64) yields (Δ_e, Δ_h). This completes the equilibrium derivation, up to the value functions (ξ_e, ξ_h).

A.3.1 Special Case: $a_e = a_h$

Under equal productivities, several simplifications can be made. First, q solves equation (44) without knowledge of the capital distribution κ. Therefore, q and $\partial_X q$ are given explicitly.

Second, if $\varsigma = 0$, we may set $\chi = 1$, without loss of generality. Indeed, in this case, we can verify by inspection that χ and κ always enter the equilibrium as the product $\alpha := \chi\kappa$ (in particular, inspect equations (66), (67), (58), (54)). Therefore, we may solve for $\alpha = \kappa$ from equation (66) under $\chi = 1$. Notice that equation (67) will then automatically be satisfied.

[13] The notation above is helpful, since it allows us to write $\|D_{\tilde{X}}\|^2 = (1 - (\chi\kappa - w)\,\partial_w \ln q)^2\,\|\sigma_R\|^2$, and simplify the expression for $\sigma_X\sigma_R$ as follows:

$$\sigma_X\sigma_R = \frac{1}{1 - (\chi\kappa - w)\,\partial_w \ln q}\begin{pmatrix} \frac{\chi\kappa - w}{1-(\chi\kappa - w)\partial_w \ln q}\|D_{\tilde{X}}\|^2 \\ \sigma_{\tilde{X}}D_{\tilde{X}} \end{pmatrix}$$

One can also verify that $\alpha = w$ if $\gamma_e = \gamma_h$ and additionally either $\psi_e = \psi_h$ or $\sigma_s = \sigma_g = 0$. If $\alpha \notin (0,1)$, we project it into $(0,1)$.

If $\varsigma > 0$, we must solve the equilibrium using equations (66) and (67) both. One simplification comes from equation (64), which shows that $\Delta_e = \underline{\chi}^{-1}\Delta_h$. Thus, either $\Delta_e = \Delta_h = 0$ (which occurs when $\chi > \underline{\chi}$) or $\Delta_e, \Delta_h > 0$ (which occurs when $\kappa < 1$).

A.4 Duffie-Epstein-Zin Preferences

For convenience, we introduce the differential operator $\mathcal{A}$, defined for any stochastic process $\{X_t\}_{t\geq 0}$ (belonging to an appropriate class of stochastic processes) as follows:

$$\mathcal{A}X_t := \lim_{\epsilon \to 0} \frac{\mathbb{E}\left[X_{t+\epsilon}|\mathcal{F}_t\right] - X_t}{\epsilon} \tag{73}$$

Assume that preferences are given by the following recursion, for $\epsilon > 0$:

$$\tilde{U}_t = \left[(1 - e^{-\rho\epsilon})c_t^{1-\psi} + \exp(-\rho\epsilon)\mathcal{R}_t\left(\tilde{U}_{t+\epsilon}\right)^{1-\psi}\right]^{\frac{1}{1-\psi}}$$

In the above, $\mathcal{R}_t\left(X_{t+\epsilon}\right) := \mathbb{E}\left[X_{t+\epsilon}^{1-\gamma}|\mathcal{F}_t\right]^{\frac{1}{1-\gamma}}$. Some manipulation leads to:

$$\tilde{U}_t^{1-\psi} - c_t^{1-\psi} = \exp(-\rho\epsilon)\left[\mathcal{R}_t\left(\tilde{U}_{t+\epsilon}\right)^{1-\psi} - c_t^{1-\psi}\right]$$

Taking limits of this expression when $\epsilon \to 0$ leads to:

$$0 = -\rho\left(\tilde{U}_t^{1-\psi} - c_t^{1-\psi}\right) + \frac{1-\psi}{1-\gamma}\tilde{U}_t^{\gamma-\psi}\mathcal{A}\tilde{U}_t^{1-\gamma}$$

The operator $\mathcal{A}$ is defined via equation (73). Making the change in variable $U_t := \frac{\tilde{U}_t^{1-\gamma}}{1-\gamma}$ and simplifying the above expression leads to:

$$0 = \rho\frac{1-\gamma}{1-\psi}U_t\left(\frac{c_t^{1-\psi}}{\left[(1-\gamma)U_t\right]^{\frac{1-\psi}{1-\gamma}}} - 1\right) + \mathcal{A}U_t \tag{74}$$

The first term in equation (74) corresponds to the utility aggregator φ defined in equation (9).

A.5 Stochastic Discount Factor

The stochastic discount factor of any agent with such recursive preferences can then be written:

$$S_t = \exp\left[\int_0^t \frac{\partial}{\partial U}\varphi\left(c_s, U_s\right) ds\right] \frac{\partial}{\partial c}\varphi\left(c_t, U_t\right)$$

Given the homogeneity properties of our model, agents' utility will be written $U_t = \frac{(n_t \xi_t)^{1-\gamma}}{1-\gamma}$, and their consumption-to-wealth ratio $c_t/n_t = \rho^{1/\psi}\xi_t^{1-1/\psi}$. Note also that the derivatives of φ are the following:

$$\begin{aligned}
\frac{\partial}{\partial c}\varphi\left(c, U\right) &= \rho c^{-\psi}\left[(1-\gamma)U\right]^{\frac{\psi-\gamma}{1-\gamma}} \\
\frac{\partial}{\partial U}\varphi\left(c, U\right) &= \rho\left(\frac{\psi-\gamma}{1-\psi}\right) c^{1-\psi}\left[(1-\gamma)U\right]^{\frac{\psi-1}{1-\gamma}} - \rho\left(\frac{1-\gamma}{1-\psi}\right)
\end{aligned}$$

Thanks to these calculations, it can be showed that:

$$S_t = \exp\left[\int_0^t \left(\left(\frac{\psi-\gamma}{1-\psi}\right)\rho^{1/\psi}\xi_s^{1-1/\psi} - \rho\left(\frac{1-\gamma}{1-\psi}\right)\right) ds\right] n_t^{-\gamma}\xi_t^{1-\gamma}$$

In the case of time- and state-separability (i.e. when $\psi = \gamma$), we obtain the familiar formula $S_t/S_0 = e^{-\rho t}\left(c_t/c_0\right)^{-\gamma}$. Remember that we have for households and experts (see (13)):

$$\begin{aligned}
\frac{dn_t}{n_t} &= \left(\mu_{n,t} - c_t/n_t\right) dt + \sigma_{n,t}\cdot dZ_t + \tilde{\sigma}_{n,t} dZ_{j,t} \\
\frac{d\xi_t}{\xi_t} &= \mu_{\xi,t} dt + \sigma_{\xi,t}\cdot dZ_t
\end{aligned}$$

In the above, $\mu_\xi := \mu_X \cdot \partial_X \ln\xi + \frac{1}{2}\left[\mathrm{tr}\left(\sigma_X' \partial_{XX'}\ln\xi\sigma_X\right) + \|\sigma_X'\partial_X \ln\xi\|^2\right]$, and $\sigma_\xi := \sigma_X'\partial_X \ln\xi$. In particular, we can obtain the vector of shadow risk prices faced by investor j via:

$$\begin{aligned}
\frac{dS_{j,t}}{S_{j,t}} - \mathbb{E}_t\left[\frac{dS_{j,t}}{S_{j,t}}\right] &= -\left[\gamma_j\sigma_{n_j,t} + (\gamma_j - 1)\sigma_{\xi_j,t}\right]\cdot dZ_t - \gamma_j\tilde{\sigma}_{n_j,t} dZ_{j,t} \\
&:= -\pi_{j,t}\cdot dZ_t - \tilde{\pi}_{j,t} dZ_{j,t}
\end{aligned}$$

In the above, the k^{th} coordinate of $\pi_{j,t}$ is the expected excess return per unit of risk exposure to the k^{th} component of Z_t that investor j gets paid. Similarly, $\tilde{\pi}_{j,t}$ is the expected excess return per unit of idiosyncratic risk that investor j gets paid.

We can compute (i) the vector of shadow risk-prices π_e and (ii) the idiosyncratic risk-price $\tilde{\pi}_e$ faced by experts, using formula (48) that gives $\sigma_{n_e,t}$ as a function of other equilibrium objects:

$$\pi_e = \gamma_e \frac{\chi\kappa}{w}\sigma_R + (\gamma_e - 1)\sigma_X' \partial_X \ln \xi_e \tag{75}$$

$$\tilde{\pi}_e = \gamma_e \frac{\chi\kappa}{w}\sqrt{\varsigma} \tag{76}$$

This means that experts' equilibrium expected excess return compensation is equal to:

$$\begin{aligned}\pi_e \cdot \sigma_R + \tilde{\pi}_e\sqrt{\varsigma} &= \gamma_e \frac{\chi\kappa}{w}\left(\|\sigma_R\|^2 + \varsigma\right) + (\gamma_e - 1)\left(\sigma_X \sigma_R\right) \cdot \partial_X \ln \xi_e \\ &= \Delta_e + \pi \cdot \sigma_R,\end{aligned}$$

where $\pi = \pi_h$ is the vector of aggregate risk prices faced by households.

Similarly, we can compute (i) the vector of shadow risk-prices π_h and (ii) the idiosyncratic risk-price $\tilde{\pi}_h$ faced by households, using formula (47) that gives $\sigma_{n_h,t}$ as a function of other equilibrium objects:

$$\pi_h = \gamma_h \frac{1-\chi\kappa}{1-w}\sigma_R + (\gamma_h - 1)\sigma_X' \partial_X \ln \xi_h \tag{77}$$

$$\tilde{\pi}_h = \gamma_h \frac{1-\kappa}{1-w}\sqrt{\varsigma} \tag{78}$$

Of course formula (77) is identical to equation (32).

A.6 Continuation Value Derivations

Assume the continuation value can be expressed as $U(n, X) = \frac{(\xi(X)n)^{1-\gamma}}{1-\gamma}$. We then have the following identities:

$$\begin{aligned}\partial_n U &= (1-\gamma)\frac{U}{n} \\ \partial_{nn} U &= -\gamma(1-\gamma)\frac{U}{n^2} \\ \partial_{nX} U &= (1-\gamma)^2 \frac{U}{n}\frac{\partial_X \xi}{\xi} = (1-\gamma)^2 \frac{U}{n}\partial_X \ln \xi \\ \partial_X U &= (1-\gamma)U\frac{\partial_X \xi}{\xi} = (1-\gamma)U\partial_X \ln \xi \\ \partial_{XX'} U &= (1-\gamma)U\frac{\partial_{XX'}\xi}{\xi} - \gamma(1-\gamma)U\frac{\partial_X \xi \partial_X \xi'}{\xi^2} = (1-\gamma)U\left[\partial_{XX'} \ln \xi + (1-\gamma)\left(\partial_X \ln \xi\right)\left(\partial_X \ln \xi\right)'\right]\end{aligned}$$

B Numerical Methods

B.1 Finite Difference Method for PDEs

The PDEs for ξ_h and ξ_e in (38)-(39) are nonlinear, and we can use an iterative procedure to solve them. These PDEs both have the general quasi-linear form

$$0 = A(x, f, \partial_x f) + \mathrm{tr}[B(x, f, \partial_x f)\partial_{xx'} f B(x, f, \partial_x f)']. \tag{79}$$

To solve, we augment (79) with a false time-derivative $\partial_t f$, known as a "false transient." Since the time-derivative appears on the right-hand-side of the PDE, the equation to solve is

$$0 = \partial_t f + A(x, f, \partial_x f) + \mathrm{tr}[B(x, f, \partial_x f)\partial_{xx'} f B(x, f, \partial_x f)']. \tag{80}$$

Thus, the original PDE (79) is the stationary solution to the augmented PDE (80), i.e., $\partial_t f = 0$ holds in (80). The following method solves (80) iteratively until $\partial_t f \approx 0$.

Step 0: *Initialization.* Form a guess for $\phi_0(x) := f(x, T)$, which is the *terminal condition.*

Step k: *Discretization, Linearization, Iteration.* Generate a grid of time points $\{T, T - \Delta t, \dots\}$ and a grid of space points $\mathcal{X}$. Given a candidate function $\phi_k(x)$ for $f(x, T - k\Delta t)$ restricted to $\mathcal{X}$, compute finite difference approximations to all derivatives. The time-derivative is approximated with the backward difference

$$\frac{\phi_k(x) - \phi_{k+1}(x)}{\Delta t} \approx \partial_t f(x, T - k\Delta t)$$

Denote the finite-difference approximations of the spatial derivatives by

$$\hat{\partial}_x \phi_k(x) \approx \partial_x f(x, T - k\Delta t)$$
$$\hat{\partial}_{xx'} \phi_k(x) \approx \partial_{xx'} f(x, T - k\Delta t).$$

Both $\hat{\partial}_x$ and $\hat{\partial}_{xx'}$ are always computed using central differences, except at the boundaries of $\mathcal{X}$, where they are computed via central differences at the nearest interior point. Using these approximations everywhere in (80), we can solve for ϕ_{k+1} given ϕ_k via

$$\phi_{k+1} = \phi_k + \Big\{A(x, \phi_k, \hat{\partial}_x \phi_k) + \mathrm{tr}[B(x, \phi_k, \hat{\partial}_x \phi_k)\hat{\partial}_{xx'} \phi_k B(x, \phi_k, \hat{\partial}_x \phi_k)']\Big\}\Delta t. \tag{81}$$

Equation (81) solves for ϕ_{k+1} explicitly, which is why this is termed the *explicit method.*

The *implicit method* applies $\hat{\partial}_x$ and $\hat{\partial}_{xx'}$ to ϕ_{k+1} rather than ϕ_k, leading to

$$\phi_{k+1} - \phi_k = \Big\{ A(x, \phi_k, \hat{\partial}_x \phi_k) + \mathrm{tr}[B(x, \phi_k, \hat{\partial}_x \phi_k) \hat{\partial}_{xx'} \phi_{k+1} B(x, \phi_k, \hat{\partial}_x \phi_k)'] \Big\} \Delta t. \tag{82}$$

Solving for ϕ_{k+1} in (82) requires solving a linear system on $\mathcal{X}$. Sometimes we use a *mixed method*, which applies $\hat{\partial}_{xx'}$ to ϕ_k for the mixed derivatives and applies $\hat{\partial}_{xx'}$ to ϕ_{k+1} for the non-mixed derivatives. Finally, sometimes we utilize the special structure of HJB equations, whereby $A(x, f, \partial_x f) = A^*(x, f, \partial_x f) + \mu_X(x, f, \partial_x f)\partial_x f$, to solve

$$\begin{aligned} \phi_{k+1} - \phi_k = \Big\{ & A^*(x, \phi_k, \hat{\partial}_x \phi_k) + \mu_X^+(x, \phi_k, \hat{\partial}_x \phi_k) \hat{\partial}_x^{(+)} \phi_{k+1} + \mu_X^-(x, \phi_k, \hat{\partial}_x \phi_k) \hat{\partial}_x^{(-)} \phi_{k+1} \\ & + \mathrm{tr}[B(x, \phi_k, \hat{\partial}_x \phi_k) \hat{\partial}_{xx'} \phi_{k+1} B(x, \phi_k, \hat{\partial}_x \phi_k)'] \Big\} \Delta t. \end{aligned} \tag{83}$$

In (83), $\hat{\partial}_x^{(+)}$ and $\hat{\partial}_x^{(-)}$ are forward and backward differences, respectively, while μ_X^+ and μ_X^- denote the positive and negative parts of μ_X, respectively. This is known as "up-winding." Using any of these methods to get ϕ_{k+1}, calculate

$$\mathrm{error}_{k+1} := \max_{x \in \mathcal{X}} \frac{|\phi_{k+1}(x) - \phi_k(x)|}{\Delta t}.$$

Given a *tolerance* for convergence tol > 0, repeat this step until $\mathrm{error}_{k+1} < \mathrm{tol}$. The function $\phi_{k+1}(x)$ is the approximate solution to (79).

B.2 Numerical Procedure

Given $\xi_e^{(n)}$ and $\xi_h^{(n)}$, we would like to update $\xi_e^{(n+1)}$ and $\xi_h^{(n+1)}$ by iterating one time-step in their PDEs. Whenever they appear, use $\xi_e = \xi_e^{(n)}$ and $\xi_h = \xi_h^{(n)}$.

1. **Inner loop: update equilibrium objects iteratively.** For any equilibrium object y, let the sequence of iterants for this inner loop be $\{\hat{y}^{(l)} : l = 0, 1, \dots\}$.[14]

   (a) If $n \geq 1$, initialize $\hat{y}^{(0)} = y^{(n-1)}$. If $n = 0$, use the guess $\hat{\kappa}^{(0)} = w$, $\hat{\chi}^{(0)} = 1$, $\hat{q}^{(0)}$ from equation (44), $\hat{\Delta}_h^{(0)} = \Delta_h^+$ from equation (65), and $\hat{\Delta}_e^{(0)} = \underline{\chi}^{-1}[\hat{\Delta}_h^{(0)} + \frac{a_e - a_h}{\hat{q}^{(0)}}]$ from equation (64).

   (b) For each $l \geq 0$, do the following:

      i. Update all other $\hat{y}^{(l)}$ objects as follows.

         A. Set $\hat{\beta}_e^{(l)} = \hat{\chi}^{(l)} \hat{\kappa}^{(l)} / w$ and $\hat{\beta}_h^{(l)} = (1 - \hat{\kappa}^{(l)})/(1 - w)$.

[14] The list of equilibrium objects is $\{q, \kappa, \chi, \sigma_K, \sigma_q, \sigma_R, \sigma_X, \pi, \Delta_e, \Delta_h, \beta_e, \beta_h, \mu_K, \mu_q, \mu_X, \mu_{R,e}, \mu_{R,h}, r\}$.

B. Set $\hat{\sigma}_K^{(l)}$, $\hat{\sigma}_q^{(l)}$, $\hat{\sigma}_R^{(l)}$, $\hat{\sigma}_X^{(l)}$, and $\hat{\pi}^{(l)}$ (in that order) using equations (1), (57), (7), (54), and (62).

C. Set $\hat{\mu}_K^{(l)}$, $\hat{\mu}_X^{(l)}$, $\hat{\mu}_q^{(l)}$, and $\hat{r}^{(l)}$ (in that order) using equations (46), (53), (56), and (61). Get $\hat{\mu}_{R,e}^{(l)}$ and $\hat{\mu}_{R,h}^{(l)}$ from equation (7).

ii. Define $\hat{\kappa}^{(l+1)} = \hat{\kappa}^{(l)} + H^{(l)} \times dt$, where dt is a small enough time-step, and $H^{(l)}$ is defined by the right-hand-side of equation (66), computed using $\chi = \underline{\chi}$ and $\hat{y}^{(l)}$ for all other objects.

iii. Denote the cubic expression in the second argument of the minimum in equation (68) by

$$F(w,\chi) := A_0(w) + A_1(w)(\chi - w) + A_2(w)(\chi - w)^2 + A_3(w)(\chi - w)^3.$$

Define $\tilde{q}$ according to equation (44) with $\kappa = 1$. Using $\tilde{q}$ and its derivatives in place of q, as well as $\kappa = 1$ and $\hat{y}^{(l)}$, compute A_0, A_1, A_2, A_3. Solve the equation $F(w,\chi) = 0$ for χ at each w. If $\varsigma = 0$, then this equation is linear and has a unique solution; otherwise, use any nonlinear solver with initial guess $\chi = \hat{\chi}^{(l)}$. Denote the solution by $\tilde{\chi}$. If $\tilde{\chi} \geq \underline{\chi}$, set $\hat{\chi}^{(l+1)} = \tilde{\chi}$. Otherwise, there are two cases:

- If $F(w,\underline{\chi}) > 0$, then set $\hat{\chi}^{(l+1)} = \underline{\chi}$.
- If $F(w,\underline{\chi}) < 0$, then set $\hat{\chi}^{(l+1)} = +\infty$ (or some very large number).

iv. Use equation (63) to solve for $\hat{\Delta}_e^{(l+1)}$, then set $\hat{\Delta}_h^{(l+1)}$ by (64). Use $\hat{\kappa}^{(l+1)}$ and $\hat{\chi}^{(l+1)}$ but $\hat{y}^{(l)}$ for everything else in this step.

v. Set $\hat{q}^{(l+1)}$ by equation (44), using $\hat{\kappa}^{(l+1)}$ and $(\xi_e^{(n)}, \xi_h^{(n)})$.

(c) Iterate on (b). When $\|\hat{\kappa}^{(l+1)} - \hat{\kappa}^{(l)}\| + \|\hat{\chi}^{(l+1)} - \hat{\chi}^{(l)}\|$ is small, stop iterating.

(d) Put $y^{(n)} = \hat{y}^{(l)}$.

2. **Outer loop: update value functions using PDEs.** Update $\xi_e^{(n+1)}, \xi_h^{(n+1)}$ by iterating one time-step in their the PDEs (39) and (38). This involves augmenting the PDEs with fictitious time derivatives, or "false transients", as discussed in Appendix B.1. If $\psi_e = 1$ or $\psi_h = 1$, replace $\frac{\psi}{1-\psi}\rho^{1/\psi}\xi^{1-1/\psi} - \frac{\rho}{1-\psi}$ by $\rho\left(\ln\rho - \ln\xi\right) - \rho$. All the coefficients in the PDEs are computed using step-n objects.

B.3 Solving Linear Systems

The following algorithm summarizes our numerical solution of PDEs (39) and (38) for $\zeta_{k\in(e,h)} := \ln \xi_{k\in(e,h)}$, based on Appendixes B.1 and B.2.

Initialization: Start with guess functions $\zeta^0_{k\in(e,h)}$.

while $\sup_x \|\zeta^t_k(x) - \zeta^{t-1}_k(x)\| > \epsilon$ ***or*** $t = 0$ **do**

(0) Update the iterator $t = t + 1$;

(1) Refresh equilibrium quantities which are functions of ζ^{t-1}_k;

(2) Update the PDE coefficients, which depend on the updated equilibrium quantities and ζ^{t-1}_k. In this process, treat the non-linear terms of ζ_k as if $\zeta_k = \zeta^{t-1}_k$. Therefore, the only terms left unknown are linear terms of ζ_k;

(3) Construct the linear systems $A^t_k \zeta^t_k = b^t_k$ as the finite difference representation of the linear PDEs from (2);

(4) Solve $A^t_k \zeta^t_k = b^t_k$ for ζ^t_k using *some linear system solver*;

end

When the algorithm finishes, we take the solutions from the final iteration in the while loop as the solutions to the PDEs.

To do step (4), which involves solving $A^t_k \zeta^t_k = b^t_k$, we have taken two distinct approaches. As a first option, we have leveraged the high-performance computing package Pardiso[15] in C++ to perform an LU decomposition of the matrix A^t_k. In what follows, we call this, including the use of the Pardiso package, the "LU approach." This has two drawbacks. First, in each iteration t, we need to perform LU decomposition for a brand new matrix, as the coefficients of the PDEs get updated in each iteration. Second, finding the LU decomposition of A^t_k is costly, because of its dimensionality (it encodes a 4-dimensional finite difference scheme, which is large, even considering its sparsity).

As a second option, we have solved $A^t_k \zeta^t_k = b^t_k$ using the conjugate gradient method, which is one example of a Krylov subspace method.

Conjugate Gradient Method. For an $n \times n$ symmetric positive definite matrix A, the conjugate gradient (CG) method solves the linear system $Ax = b$ by minimizing the quadratic form:

$$\min \; f(x) = \frac{1}{2} x^T A x - x^T b$$

The first order condition of this minimization problem is

$$\partial_x f(x) = Ax - b = 0$$

Therefore, if A is positive definite, solving the linear system is equivalent to finding the min-

[15] https://www.pardiso-project.org/

imum of the quadratic form. The matrix from our finite difference scheme is not symmetric, but we can do the following transformation: $\tilde{A} = A^T A$ and $\tilde{b} = A^T b$. Applying the CG method to $\tilde{A}x = \tilde{b}$ yields the same results as long as A is invertible.

The CG method is an iterative method that solves the minimization problem above via an algorithm of the form:

$$x_{k+1} = x_k + \alpha_k p_k$$

The search directions $\{p_k\}_{k \leq n}$ are orthogonal under the scalar product induced by the matrix A (i.e. they form an "A-conjugate set"), and the step length $\{\alpha_k\}_{k \leq n}$ are computed so as to minimize the quadratic form applied to $x_k + \alpha_k p_k$. One theoretical result of such iteration is that the CG method converges in at most n steps, where n is the dimension of the linear system. Another appealing feature is that CG builds the directions p_k iteratively by only knowing the previous direction p_{k-1} and the gradient of the quadratic form evaluated at x_k.

Incorporating CG into Model Solution. As an iterative method, CG performs better with a more accurate initial guess for ζ_k^t. Since our model solution method is already iterative and (hopefully converges), we propose to use ζ_k^{t-1} as the starting point, instead of an arbitrary point. As the sequence $\{\zeta_k^t\}_{\{t \geq 0\}}$ converges, its elements become closer together. Intuitively, using this information allows CG to find the minimizer in a smaller amount of time as the number of iterations progresses. In what follows, we call this the "smart guess approach."

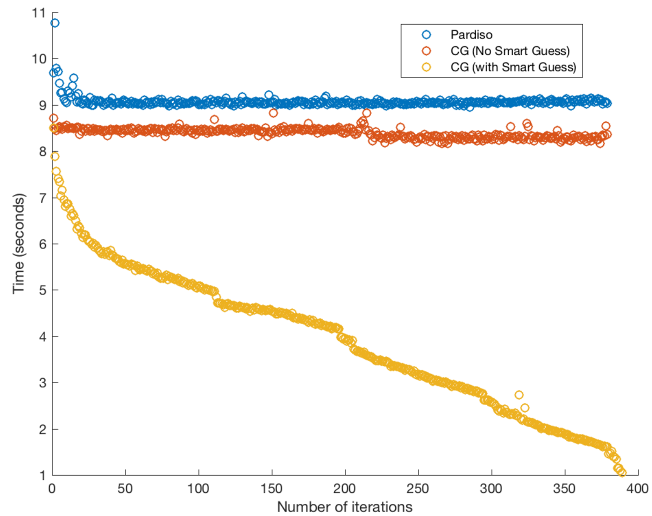
Test: CG vs LU Decomposition. For simplicity, we test the case where $\gamma = \psi = 1$ for both agents (logarithmic utility). The model is solved in three dimensions with the grid size $100 \times 50 \times 50$ (100 points in the w direction). We implemented three approaches: LU with 28 cores, CG without the smart guess approach (i.e. with an arbitrary starting point in each inner loop), and CG with the smart guess approach. We only equip the CG approaches with 1 core, since the sparse matrices do not contain that many non-zeroes to justify parallelization. If anything, this "unfair" treatment strengthens the case for CG.

As shown in figure 11, CG is faster. Furthermore, the speed gains accruing to the smart guess method magnify as the iteration number becomes larger, intuitively because the guesses are becoming more and more accurate as $\{\zeta_k^t\}$ converges. Finally, this is only a naive implementation of CG. Areas of improvement include:

1. Unlike LU, CG finds an approximate solution, which can be leveraged to make speed gains. In particular, in each CG call, we need to set a tolerance for CG to stop. We can use higher tolerance in the beginning and reduce the tolerance gradually as the outer loop proceeds;

2. The package we have currently used is not designed for CG. We can look for dedicated CG packages;

3. As the dimensions approach 4, we can start using parallel cores.



(a) Time taken per outer loop iteration

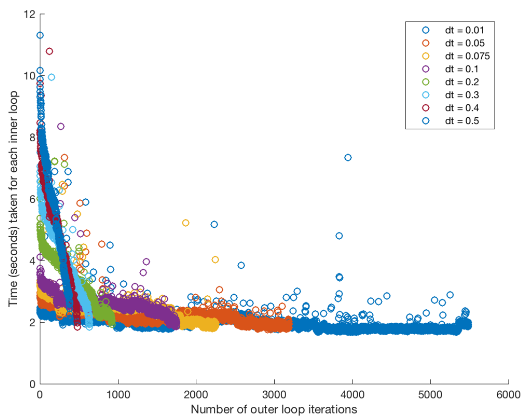
Approach	Total Time (minutes)
Pardiso (LU Decomposition)	57
CG (Without Smart Guess)	53
CG (With Smart Guess)	25

(b) Total time taken to solve model

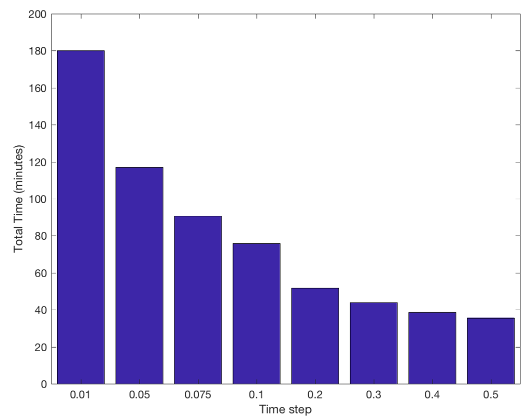
Figure 11: Time Performance for CG vs LU Decomposition (implemented by Pardiso)

Test: Role of Time Step. The time step (Δt), used in iterating on the PDEs for ζ_k, is an important numerical parameter. With CG, Δt captures a trade-off between speed of convergence in the VFI "outer loop" versus the CG "inner loop". A higher Δt reduces the number of value function iterations (VFI) for $\{\zeta_k^t\}$ to converge in the outer loop. On the other hand, higher Δt increases the number of CG inner loop iterations, as the initial guesses for ζ_t^k are less accurate. Figure 12 illustrates this trade-off.

Although the highest $\Delta t = 0.5$ requires significantly more time in the beginning, it quickly finds the right solution and the number of inner loop iterations decreases very fast (perhaps aided by the smart guess approach). With $\Delta t = 0.01$, the algorithm requires significantly more outer loop iterations and the total time taken is longer. This test shows that, in this case, reducing the number of iterations (higher Δt) dominates very accurate guesses at all iterations (lower Δt).



(a) Time taken per outer loop iteration



(b) Total time taken to solve model

Figure 12: Role of Time Step

C Shock Elasticities

Some recent papers have characterized shock-exposure and shock-price elasticities, arguing that they is an alternative and useful way to depict asset prices in dynamic models.[16] The following discussion provides a short overview.

Setup and Definition. In the class of models we consider, there will always be a n-dimensional state variable X_t following a diffusion

$$dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dZ_t \tag{84}$$

There will also be an equilibrium stochastic discount factor S_t, which follows

$$d\log S_t = \mu_S(X_t)dt + \sigma_S(X_t) \cdot dZ_t. \tag{85}$$

Finally, any cash flow process $\{G_t\}$ can be constructed from the Markov state $\{X_t\}$ as

$$d\log G_t = \mu_G(X_t)dt + \sigma_G(X_t) \cdot dZ_t. \tag{86}$$

Assume this cash flow $\{G_t\}$ is priced by the SDF $\{S_t\}$. Given these processes, we can construct shock elasticities as follows. Define $\{M_t\}$ be a logarithmic process analogous to $\{G_t\}$ and $\{S_t\}$:

$$d\log M_t = \mu_M(X_t)dt + \sigma_M(X_t) \cdot dZ_t.$$

Define the exponential martingale $H_s^\nu := \exp\big(\int_0^s \nu(X_u)\cdot dZ_u - \frac{1}{2}\int_0^s |\nu(X_u)|^2 du\big)$, where $\nu(x) \in \mathbb{R}^d$ and $\|\nu(x)\| = 1$. Then, define the *shock elasticity* for process M at horizon t to be

$$\varepsilon_M(t,x) := \lim_{s \searrow 0} \frac{1}{s} \log \mathbb{E}\Big[\Big(\frac{M_t}{M_0}\Big) H_s^\nu \mid X_0 = x\Big], \tag{87}$$

By Girsanov's theorem, H_s^ν acts to alter the distribution of shocks to which M is exposed between times $[0,s]$. By altering the distribution of shocks, $\varepsilon_M(t,x)$ measures a type of non-linear impulse response, tracing the effect of this altered distribution over a horizon t.[17]

Interpretation as an Increase in Exposure. Equivalently, H_s^ν acts to perturb the shock

[16] See Borovička et al. (2011), Hansen (2012), Hansen (2013). An accessible review treatment is provided in the handbook chapter Borovička and Hansen (2016).

[17] This logic is made precise in Borovička, Hansen, and Scheinkman (2014).

exposure of M near time 0. To see this equivalence, look at

$$\log \frac{M_t}{M_0} H_s^\nu = \int_0^s \left[\mu_M(X_u) - \frac{1}{2}\nu(X_u)\right] du + \int_0^s \left[\sigma_M(X_u) + \nu(X_u)\right] \cdot dZ_u \\ + \int_s^t \mu_M(X_u) du + \int_s^t \sigma_M(X_u) \cdot dZ_u.$$

On the interval $[0, s]$, MH^ν has perturbed exposure $\sigma_M(X_u) + \nu(X_u)$ to the Brownian shock dZ_u. For example, when $M = G$, we can think of $M_t H_s^\nu$ as a perturbed cash flow, which has a vector ν of additional shock exposures near time 0. Then the function $\varepsilon_M(t, x)$ traces out the expected effect of this instantaneous increase in shock exposure over a horizon t. The fact that $\|\nu(x)\| = 1$ justifies the use of the term "elasticity". We will typically take ν to be a coordinate vector to interpret ε_M as the response of M to a particular shock.

Asset-Pricing. The *shock-exposure elasticity* is defined by ε_G and the *shock-price elasticity* is defined by $\varepsilon_G - \varepsilon_{SG}$. The shock-exposure elasticity answers the question: how sensitive is the expected cash flow G_t to an increase in its risk exposure at time 0?

The shock-price elasticity is slightly more nuanced. Because $\mathbb{E}[S_t G_t]$ is the scaled price of the cash flow G_t, and GH^ν is the perturbed cash flow, ε_{SG} is the sensitivity of the price to an increase in the risk exposure of G_t. Since $\log(\mathbb{E}[G_t]\mathbb{E}[S_t]/\mathbb{E}[S_t G_t])$ is a log expected excess return, shock-price elasticities answer the question: how much are expected excess returns required to increase with an increase in the exposure of G_t to a particular shock at time 0? It is in units of Sharpe ratios, or risk prices, because H^ν delivers a unit standard deviation increase in the risk exposure of G_t. Thus, the shock-price elasticity is often interpreted as a *term structure of risk prices.*

Computation. The shock elasticities are computed by applying Malliavin calculus, which is beyond the scope of this Appendix. The result is (see the papers cited above)

$$\varepsilon_M(t, x) = \nu(x) \cdot \left\{\sigma_M(x) + \sigma_X(x) \cdot \frac{\partial}{\partial x} \log \mathbb{E}\left[\left(\frac{M_t}{M_0}\right) \mid X_0 = x\right]\right\}. \tag{88}$$

To compute each of the conditional expectations in (88) numerically, we solve a PDE derived as follows. Define $f_M(t, x) := \mathbb{E}[\frac{M_t}{M_0} f_M(0, X_t) \mid X_0 = x]$. Then, using the law of iterated expectations, followed by the definition of f_M, we have $f_M(t, x) = \mathbb{E}[\frac{M_u}{M_0} \mathbb{E}[\frac{M_t}{M_u} f_M(0, X_t) \mid X_u] \mid X_0 = x] = \mathbb{E}[\frac{M_u}{M_0} f_M(t-u, X_u) \mid X_0 = x]$. Hence, $\{M_t f_M(T-t, X_t)\}_{t \in [0,T]}$ is a martingale and must have zero drift. Applying Itô's formula gives a PDE for f_M in (t, x), i.e.,

$$0 = -\frac{\partial f_M}{\partial t} + \left(\mu_M + \frac{1}{2}\|\sigma_M\|^2\right) f_M + \left(\mu_X + \sigma_M \cdot \sigma_X\right) \frac{\partial f_M}{\partial x} + \frac{1}{2}\|\sigma_X\|^2 \frac{\partial^2 f_M}{\partial x^2}. \tag{89}$$

The initial condition is $f_M(0,x) \equiv 1$, which allows us to recover the desired conditional expectation. We obtain $\varepsilon_M(t,x)$ by numerically differentiating $f_M(t,x)$ and substituting it into (88). Note also that the term structure of interest rates can be obtained by solving PDE (89) with $M = S$ and taking $r(t,x) := -\frac{1}{t} \log f_S(t,x)$.

We solve PDE (89) subject to boundary conditions at the boundaries of the domain of X_t. These typically depend on the nature of the model, i.e., (μ_X, σ_X), as well as the process (μ_M, σ_M). In models we consider, boundaries are inaccessible by X_t, so we impose zero first derivatives at those boundaries.

Alternative Shock Elasticities. There is also a second type of shock elasticity, which differs conceptually from the first type. While $\varepsilon_M(t,x)$ measures the expected response of M_t to a shock at time 0, we could also compute the expected response of M_t to a shock at the same time t. Denote this elasticity by $\tilde{\varepsilon}_M(t,x)$, which is computed by replacing H_s^ν in (87) by $\tilde{H}_s^\nu := \exp(\int_t^{t+s} \nu(X_u) \cdot dZ_u - \frac{1}{2}\int_t^{t+s} |\nu(X_u)|^2 du)$. Because $\tilde{H}^\nu$ is also a martingale perturbation, this alternative shock elasticity shares some interpretations with the benchmark shock elasticities, except the shock is presumed to impact M at some future time.

It turns out that we can compute this alternative shock elasticity via

$$\tilde{\varepsilon}_M(t,x) = \nu(x) \cdot \frac{\mathbb{E}[\frac{M_t}{M_0}\sigma_M(X_t) \mid X_0 = x]}{\mathbb{E}[\frac{M_t}{M_0} \mid X_0 = x]}, \tag{90}$$

which requires solving the PDE (89) twice with initial conditions $f_M(0,x) \equiv \sigma_M(x)$ and $f_M(0,x) \equiv 1$, then taking the ratio to get $\tilde{\varepsilon}_M$.

D Long Run Risk Model with Production

In this case, the aggregate state vector X_t is simply (g_t, s_t). Thus, the drift vector $\mu_X(X)$ and the volatility matrix $\sigma_X(X)$ are exogenously specified. We assume complete markets. We show that if we know $\pi(X), r(X)$, we can compute the (scaled) value function $\xi(X)$. Indeed, the PDE that ξ solves is the following:

$$0 = \frac{\psi}{1-\psi}\rho^{1/\psi}\xi^{1-1/\psi} - \frac{\rho}{1-\psi} + r + \frac{\|\pi\|^2}{2\gamma} + \left[\mu_X + \frac{1-\gamma}{\gamma}\sigma_X\pi\right]\cdot\partial_X \ln\xi + \frac{1}{2}\left[\mathrm{tr}(\sigma_X'\partial_{XX'}\ln\xi\sigma_X) + \frac{1-\gamma}{\gamma}\|\sigma_X'\partial_X\ln\xi\|^2\right] \quad (91)$$

We then show that if we know $\xi(X)$, we can compute all the equilibrium objects $\pi(X), r(X), q(X)$. First, the consumption market clearing gives us q:

$$\rho^{1/\psi}\xi^{1-1/\psi} = \frac{a - \Phi(\iota(q))}{q}$$

In the above, $\iota(q) = (\Phi')^{-1}(q)$. Using our specification for the adjustment cost function Φ, we obtain:

$$q = \frac{a + 1/\phi}{\rho^{1/\psi}\xi^{1-1/\psi} + 1/\phi}$$

Second, equalizing the drifts and volatilities of $qK = N$ leads to:

$$\mu_q + \mu_K + \sigma_q\cdot\sigma_K = \mu_n - \hat{c}$$
$$\sigma_q + \sigma_K = \sigma_n$$

Remember that we have:

$$\mu_n = r + \sigma_n\cdot\pi$$
$$\sigma_n = \frac{\pi}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_X'\partial_X\ln\xi$$
$$\hat{c} = \rho^{1/\psi}\xi^{1-1/\psi}$$

Finally, remember that:

$$\mu_K = \iota(q) + g - \delta$$
$$\sigma_K = \sqrt{s}\sigma$$

Matching the volatility terms leads to:

$$\sigma_X' \partial_X \ln q + \sqrt{s}\sigma = \frac{\pi}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_X' \partial_X \ln \xi$$
$$\Rightarrow \pi = \gamma \sigma_X' \partial_X \ln q + (\gamma - 1)\sigma_X' \partial_X \ln \xi + \gamma \sqrt{s}\sigma$$

Matching the drift terms leads to the following expression for r:

$$r = \mu_X \cdot \partial_X \ln q + \frac{1}{2}\left[\operatorname{tr}\left(\sigma_X' \partial_{XX'} \ln q \sigma_X\right) + |\sigma_X' \partial_X \ln q|^2\right] + \iota(q) + g - \delta + \sqrt{s}\sigma \cdot (\sigma_X' \partial_X \ln q)$$
$$- \frac{\|\pi\|^2}{\gamma} + \frac{\gamma - 1}{\gamma}\pi \cdot (\sigma_X' \partial_X \ln \xi) + \rho^{1/\psi}\xi^{1-1/\psi}$$

Reinjecting into (91), it can then be verified that the PDE that ξ satisfies is the following:

$$0 = \frac{\rho^{1/\psi}\xi^{1-1/\psi} - \rho}{1-\psi} + \left[\mu_X + (1-\gamma)\sqrt{s}\sigma\sigma_X\right] \cdot \partial_X \ln (q\xi) + \frac{1}{2}\operatorname{tr}\left(\sigma_X' \partial_{XX'} \ln (q\xi)\, \sigma_X\right)$$
$$+ \frac{1-\gamma}{2}\|\sigma_X' \partial_X \ln (q\xi)\,\|^2 + \iota(q) + g - \delta - \frac{\gamma}{2}\|\sigma\|^2 s$$

In the unitary IES case, the capital price is constant, equal to $q = \frac{a+1/\phi}{\rho+1/\phi}$. The risk price and risk free rates take the familiar form:

$$\pi = (\gamma - 1)\sigma_X' \partial_X \ln \xi + \gamma \sqrt{s}\sigma$$
$$r = \rho + \iota(q) + g - \delta - \sqrt{s}\sigma \cdot \pi$$

Note that in that case, the PDE satisfied by ξ simplifies to:

$$\rho \ln \rho - \rho \ln \xi + \iota(q) + g - \delta - \frac{\gamma}{2}\|\sigma\|^2 s + \left[\mu_X + (1-\gamma)\sqrt{s}\sigma_X \sigma\right] \cdot \partial_X \ln \xi$$
$$+ \frac{1}{2}\operatorname{tr}(\sigma_X' \partial_{XX'} \ln \xi \sigma_X) + \frac{1-\gamma}{2}\|\sigma_X' \partial_X \ln \xi\|^2 = 0$$

Remember that the drift vector and volatility matrix take the following form:

$$\mu_X = \begin{pmatrix} \lambda_g(\overline{g} - g) \\ \lambda_s(\overline{s} - s) \end{pmatrix} \tag{92}$$
$$\sigma_X = \sqrt{s}\begin{pmatrix} \sigma_g' \\ \sigma_s' \end{pmatrix} \tag{93}$$

Then guess that $\ln \xi(g, s) = \alpha_0 + \alpha_g g + \alpha_s s$, reinject to find 3 equations in 3 unknown $\alpha_0, \alpha_g, \alpha_s$:

$$\rho \ln \rho - \rho\alpha_0 + \iota(q) - \delta + \lambda_g \bar{g} \alpha_g + \lambda_s \bar{s} \alpha_s = 0$$
$$-\rho\alpha_g + 1 - \lambda_g \alpha_g = 0$$
$$-\rho\alpha_s - \frac{\gamma}{2}\|\sigma\|^2 + (1-\gamma)\sigma \cdot [\alpha_g \sigma_g + \alpha_s \sigma_s] - \lambda_s \alpha_s + \frac{1-\gamma}{2}\|\alpha_g \sigma_g + \alpha_s \sigma_s\|^2 = 0$$

When $\gamma \neq 1$, α_s is the root of a quadratic equation:

$$\frac{1-\gamma}{2}\|\sigma_s\|^2 \alpha_s^2 + \left[(1-\gamma)\,\sigma_s \cdot (\alpha_g \sigma_g + \sigma) - (\rho + \lambda_s)\right]\alpha_s + \frac{1-\gamma}{2}\|\alpha_g \sigma_g + \sigma\|^2 - \frac{\|\sigma\|^2}{2} = 0$$

We are interested in the root α_s of this quadratic equation such that the implied long-run risk-neutral measure induces stochastic stability. As argued by Hansen and Scheinkman (2009), this is the right-most zero of the quadratic equation above. We obtain the following:

$$\alpha_g = \frac{1}{\rho + \lambda_g} \tag{94}$$

$$\alpha_s = \left[\frac{(\gamma-1)\sigma_s \cdot (\alpha_g \sigma_g + \sigma) + \rho + \lambda_s}{(\gamma-1)\|\sigma_s\|^2}\right]\left[\sqrt{1 - \frac{\|\alpha_g \sigma_g + \sigma\|^2 - \frac{\|\sigma\|^2}{(\gamma-1)}}{\left(\frac{(\gamma-1)\sigma_s \cdot (\alpha_g \sigma_g + \sigma) + \rho + \lambda_s}{(\gamma-1)\|\sigma_s\|}\right)^2}} - 1\right] \tag{95}$$

$$\alpha_0 = \frac{\rho \ln \rho + \iota(q) - \delta + \lambda_g \bar{g} \alpha_g + \lambda_s \bar{s} \alpha_s}{\rho} \tag{96}$$

If $\gamma = 1$, the coefficient α_s is instead equal to $\alpha_s = \frac{-\|\sigma\|^2}{2(\rho + \lambda_s)}$. In this particular model set-up, risk-prices and risk-free rates take the following form:

$$\pi = \sqrt{s}\left[(\gamma - 1)\left(\alpha_g \sigma_g + \alpha_s \sigma_s\right) + \gamma\sigma\right] \tag{97}$$

$$r = \rho + \iota(q) + g - \delta - s\left[\gamma\|\sigma\|^2 + (\gamma-1)\sigma \cdot (\alpha_g \sigma_g + \alpha_s \sigma_s)\right] \tag{98}$$

The expected excess return on capital is then the following:

$$\begin{aligned}\mathbb{E}_t\left[dR_t - r_t dt\right] &= \sigma_{R,t} \cdot \pi_t \\ &= s_t\left[(\gamma-1)\left(\alpha_g \sigma_g + \alpha_s \sigma_s\right) + \gamma\sigma\right] \cdot \sigma\end{aligned}$$

With $\psi = 1$, the consumption-capital ratio is constant, so consumption dynamics $d\log C_t = \mu_{C,t}dt + \sigma_{C,t} \cdot dZ_t$ are given by

$$\begin{aligned}
\mu_C &= \iota(q) - \delta + g - \frac{1}{2}s\|\sigma\|^2 := \bar{\mu}_{C_0} + \bar{\mu}_{C_g}(g - \bar{g}) + \bar{\mu}_{C_s}(s - \bar{s}) \\
\sigma_C &= \sqrt{s}\sigma := \sqrt{s}\bar{\sigma}_C
\end{aligned}$$

Similarly, if S_t is the stochastic discount factor, $d\log S_t = \mu_{S,t}dt + \sigma_{S,t} \cdot dZ_t$ are given by

$$\begin{aligned}
\mu_S &= -r - \frac{1}{2}\|\pi\|^2 := \bar{\mu}_{S_0} + \bar{\mu}_{S_g}(g - \bar{g}) + \bar{\mu}_{S_s}(s - \bar{s}) \\
\sigma_S &= -\pi := \sqrt{s}\bar{\sigma}_S
\end{aligned}$$

In the above, we have introduced the constants $\bar{\mu}_{S_0}, \bar{\mu}_{C_0}, \bar{\mu}_{S_g}, \bar{\mu}_{C_g}, \bar{\mu}_{S_s}, \bar{\mu}_{C_s}, \bar{\sigma}_S$:

$$\begin{aligned}
\bar{\mu}_{S_g} &:= -1 \\
\bar{\mu}_{C_g} &:= 1 \\
\bar{\mu}_{S_s} &:= -\frac{1}{2}\left[\left[(\gamma - 1)\left(\alpha_g\sigma_g + \alpha_s\sigma_s + \sigma\right)\right]^2 - \|\sigma\|^2\right] \\
\bar{\mu}_{C_s} &:= -\frac{1}{2}\|\sigma\|^2 \\
\bar{\mu}_{S_0} &:= -\rho - \iota(q) + \delta + \bar{\mu}_{S_g}\bar{g} + \bar{\mu}_{S_s}\bar{s} \\
\bar{\mu}_{C_0} &:= \iota(q) - \delta + \bar{\mu}_{C_g}\bar{g} + \bar{\mu}_{C_s}\bar{s} \\
\bar{\sigma}_S &:= -\left[(\gamma - 1)\left(\alpha_g\sigma_g + \alpha_s\sigma_s\right) + \gamma\sigma\right] \\
\bar{\sigma}_C &:= \sigma
\end{aligned}$$