Asset Pricing with Heterogeneous Agents and Long-Run Risk*

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Abstract

This paper examines the effect of agent belief heterogeneity on long-run risk models. We find that for the long-run risk explanation to explain the equity premium, it is insufficient for long-run risk to merely *exist*: agents must *all agree* that it exists. Agents who believe in a lower persistence level dominate the economy rather quickly, even if their belief is wrong. This drives the equity premium down below the level observed in the data. On the positive side, we show that belief heterogeneity can generate significant excess volatility, which explains the large volatility of the price–dividend ratio observed in the data.

Keywords: asset pricing, long-run risk, recursive preferences, heterogeneous agents.

JEL codes: G11, G12.

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1 Introduction

This paper examines the effect of agent belief heterogeneity on consumption-based asset-pricing models with long-run risk. We document that for the long-run risk explanation to adequately explain the equity premium, it is not sufficient for long-run risk to merely *exist*: agents must *all agree* that it exists. Agents who believe in a lower persistence level come to dominate the economy rather quickly, even if their belief is wrong but not too far off the truth. This drives the equity premium down below the level observed in the data. On the positive side, we show that for intermediate levels of belief heterogeneity both agents survive in the long run and the resulting economy can generate significant excess volatility. In fact, such a model specification helps to explain, for example, the large volatility of the price–dividend ratio observed in the data.

The Bansal–Yaron long-run risk model (Bansal and Yaron (2004)) has emerged as perhaps the premier consumption-based asset-pricing model. It can generate many of the features of aggregate stock prices that have long been considered puzzles. The model generates a high equity premium by combining two mechanisms—investors with a taste for the early resolution of uncertainty, and very persistent shocks to the growth rate of consumption. For long-run risk to generate a high equity premium, the level of persistence must be very close to a unit root. The amount of persistence in the data is very difficult to measure, and arguments for a range of estimates have appeared in the literature (Bansal, Kiku, and Yaron (2016), Schorfheide, Song, and Yaron (2018), or Grammig and Schaub (2014)). This literature suggests that there is considerable scope for disagreement over the true value.

In this paper, we consider the consequences if the agents themselves disagree about its persistence. If the disagreement is sufficiently large, then agents whose beliefs are more correct dominate the economy, in accordance with the market selection hypothesis of Alchian (1950) and Friedman (1953). As we shrink the difference, the situation changes dramatically. Investors who believe in a lower persistence level or—put differently—who are more skeptical about the presence of long-run risks, not only survive in the long run but, in fact, enjoy a larger share of consumption, even if their beliefs are wrong.

For small levels of disagreement, we find that investors who are more skeptical about the presence of long-run risks accumulate wealth on average. Even if they initially hold only a very small consumption share, their share increases rapidly over time. As small differences in beliefs about the long-run risk process have large effects on asset prices, we report a drop in the equity premium of 2% within a century for our baseline calibration.

For moderate levels of disagreement, the initial increase in the consumption share of the skeptical investors is even stronger, irrespective of whether their beliefs are correct or not. If they hold the wrong beliefs, both agents survive, and asset prices become very volatile as the wealth distribution shifts over time. Thus, agent heterogeneity itself can serve as a source of endogenous asset-pricing volatility. However, the drop in the equity premium becomes even more pronounced due to the fast initial increase in the consumption share of the skeptical investors. For the calibration in this paper, we report a decrease in the premium of more than 3.5% within 100 years.

While differences in beliefs potentially undermine the ability of long-run risk to explain the equity premium, they significantly help in explaining the volatility figures. Beeler and Campbell (2012) show that the long-run risk models of Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012) cannot explain the large volatility of the price–dividend ratio observed in the data (a value of 0.45 compared to 0.18 in the models). Moderate differences in beliefs about long-run risk can generate large shifts in the wealth distribution over time when both agents survive in the long run. These shifts in turn increase the volatility of the price–dividend ratio as the impact of the different agents on asset prices varies over time. We find that a moderate difference can generate significant excess volatility close to the values observed in the data. This result also gives a model-based explanation for the empirical findings of Carlin, Longstaff, and Matoba (2014), who use data from the mortgage-backed security market and show that higher disagreement leads to higher volatility. They also show that, as in our model, disagreement is time-varying and correlated with macroeconomic variables.

These results may seem surprising because it is well established that for agents with constant relative risk aversion (CRRA) in the long run the agents with correct beliefs (Sandroni (2000), Blume and Easley (2006), Yan (2008)) always come to dominate the economy, no matter how large or small the difference in beliefs. This analysis breaks down once you allow agents to have preferences for the early or late resolution of risk (Borovička (2015)), which allows agents with incorrect beliefs to survive and even drive out agents with correct beliefs.

Our results are complementary to the findings of Collin-Dufresne, Johannes, and Lochstoer (2016b) and Bidder and Dew-Becker (2016), which show that the asset-pricing implications of long-run risk can emerge endogenously from parameter uncertainty, even without long-run risk being present. Collin-Dufresne, Johannes, and Lochstoer (2016b) show that if investors learn the growth rate from the data, then innovations to expectations of growth rates are permanent. Agents then price in the risk from this permanent shock to their expected growth rates. Bidder and Dew-Becker (2016) show that ambiguity-averse investors will price in long-run risk if they cannot rule it out a priori. In our setup, neither investor suffers from model uncertainty, but despite this difference a clear picture of the effect of long-run risk emerges.

Also relevant for our work is the paper by Andrei, Carlin, and Hasler (2016). While in the present paper the agents agree to disagree about the long-run risks in the economy, Andrei, Carlin, and Hasler (2016) provide an explanation of how this disagreement can arise from model uncertainty as market participants calibrate their models differently. They find that uncertainty about long-run risks can explain many stylized facts of stock return volatilities, such as large volatilities during recessions and booms and persistent volatility clustering. Andrei, Hasler, and Jeanneret (2017) show how model uncertainty can lead to long-run-risk-like behavior in the presence of a noisy signal of the growth rate.

Related Literature. The study of agent belief heterogeneity begins with the market selection hypothesis of Alchian (1950) and Friedman (1953). By analogy with natural selection, the market selection hypothesis states that agents with systematically wrong beliefs will eventually be driven out of the market. The influence of agent heterogeneity on market outcomes under the standard assumption of discounted expected utility is well understood, and consistent with market selection. Sandroni (2000) and Blume and Easley (2006) find strong support for this hypothesis under the assumption of time-separable preferences in an economy without growth. Yan (2008) and Cvitanić, Jouini, Malamud, and Napp (2012) analyze the survival of investors in a continuous-time framework where there are not only differences in the beliefs but also potentially differences in the utility parameters of the investors. They show that it is always the investor with the lowest survival index[1] who survives in the long run. However, the 'long run' can indeed be very long and, therefore, irrational investors can have significant effects on asset prices even under the assumption of discounted expected utility. David (2008) considers a similar model setup, in which both agents have distorted estimates for the mean growth rate of the economy, and shows that—as agents with lower risk aversion undertake more aggressive trading strategies—the equity premium increases the lower the risk aversion is. Chen, Joslin, and Tran (2012) analyze how differences in the beliefs about the probability of disasters affect asset prices. They show that even if there is only a small fraction of investors who are optimistic about disasters, this fraction sells insurance for the disaster states and so eliminates most of the risk premium associated with disaster risk. Bhamra and Uppal (2014) consider the case of habit utility.

For recursive utility, this qualitative behavior changes fundamentally. However, there has

[1] Yan (2008) shows that the survival index increases with the belief distortion, risk aversion, and subjective time discount rate of the investor.

been less research in this area, as solving such models is anything but trivial. Lucas and Stokey (1984) observe in the deterministic case that the problem of finding all Pareto-optimal allocations can be made recursive if we allow the weights in the social welfare function to be time-varying. This approach is extended by Kan (1995) to the stochastic case with finite state spaces. Anderson (2005) develops an extensive theory for the special case of risk-sensitive preferences and finite state spaces, and finds first-order conditions similar to those we use below. In particular, he shows how to characterize the equilibrium by a single value function instead of one value function for each agent. Collin-Dufresne, Johannes, and Lochstoer (2015) derive similar first-order conditions to ours for recursive utility by equating marginal utilities, but use a different procedure to solve for their allocations. Duffie, Geoffard, and Skiadas (1994) formulate the problem in continuous time, while Dumas, Uppal, and Wang (2000) reformulate it in terms of variational utility. Borovička (2015) uses this formulation to explore the question of the survival of agents with recursive utility in continuous time, and shows that agents with fundamentally wrong beliefs can survive or even dominate.[2] So, inferences about market selection and equilibrium outcomes fundamentally differ under the assumption of general recursive utility compared to the special case of standard time-separable preferences. While Borovička (2015) concentrates on the special case of i.i.d. consumption growth, Branger, Dumitrescu, Ivanova, and Schlag (2011) generalize the results to a model with long-run risks as a state variable.

However, most papers with heterogeneous investors and recursive preferences consider only an i.i.d. process for consumption growth. For example, Gârleanu and Panageas (2015) analyze the influence of heterogeneity in the preference parameters on asset prices in a two-agent OLG economy. Roche (2011) considers a model in which the heterogeneous investors can only invest in a stock but there is no risk-free bond. Hence, as there is no savings trade-off, the impact of recursive preferences on equilibrium outcomes will be quite different.

Exceptions that relax the i.i.d. assumptions include, for example, the papers by Branger, Konermann, and Schlag (2015) or Collin-Dufresne, Johannes, and Lochstoer (2016a). Both papers reexamine the influence of belief differences regarding disaster risk with Epstein–Zin instead of CRRA preferences as in Chen, Joslin, and Tran (2012). Branger, Konermann, and Schlag (2015) provide evidence that the influence of investors with more optimistic beliefs about disasters is less profound when the disaster occurs to the growth rate of consumption and investors have recursive preferences. Collin-Dufresne, Johannes, and Lochstoer (2016a)

[2] Borovička (2015) describes four channels that affect equilibrium outcomes. We examine these channels in more detail in Section 4.1 and show how they affect equilibrium outcomes in the asset-pricing model considered in this paper.

make a similar claim but for a different reason. They show that if the investors can learn about the probability of disaster and if they have recursive preferences, the impact of the optimistic investor on asset prices decreases. Optimists are uncertain about the probability of disaster and hence will provide less insurance to the pessimistic investors. Collin-Dufresne, Johannes, and Lochstoer (2016a) use an OLG model with two generations to model optimists and pessimists. Hence—in contrast to the results in the present study—the consumption shares of the investors are fixed and the increasing influence of optimistic agents over time is not captured.

In a different direction, Epstein, Farhi, and Strzalecki (2014) argue that an Epstein–Zin investor dislikes long-run risk to the extent that he or she would pay a substantial premium to get rid of it. In a model with two agents, the agent who believes that risk is longer term than the other is willing to pay an insurance premium to the other agent to hedge against long-run risk.

The remainder of the paper is organized as follows; In Section 2 we describe the general asset-pricing model with heterogeneous investors and recursive preferences. Section 3 describes the model specification with long-run risks and provides a justification for persistent belief differences. In Section 4 we present results for the baseline model and explain the economic mechanisms that generate these results. In Section 5 we relax the assumption of identical preferences and analyze the influence of differences in the preference parameters on equilibrium outcomes, both for economies with identical and with different beliefs. Section 6 concludes. Online appendices presenting the proofs of all theoretical results, a description of the numerical solution method, and additional results complete the paper.

2 Theoretical Framework

We consider a standard infinite-horizon discrete-time endowment economy with a finite number of heterogeneous agents. Agents can differ with respect to both their utility functions and their subjective beliefs. We restrict our attention to the complete-markets setting, which allows us to reformulate the problem as a social planner's problem. Here we run into a critical difference to the representative-agent problem—even for a Markov economy, equilibrium allocations are no longer required to be functions of the exogenous state alone. This feature defeats most of the approaches to solving for equilibrium in an infinite-horizon asset-pricing model.

This failure of recursiveness occurs for essentially economic reasons—even if aggregate consumption does not contain a trend, the individual consumption allocations can do so. For example, Blume and Easley (2006) show that if agents have different beliefs, then the individual consumption of an agent with wrong beliefs will trend down over time. Yan (2008) shows that in an economy with growth and with agents with differing risk aversion, the relative consumption of the more risk-averse agent tends downward.

We present a reformulation of the first-order conditions for equilibrium that is recursive. This reformulation involves introducing new endogenous state variables. Interestingly, these state variables have a clear interpretation in terms of time-varying weights in the social planner's problem. The weights capture the relative trend in an agent's consumption—an agent who has a declining share of consumption will have a declining weight.

2.1 The Heterogeneous-Agents Economy

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. Let y_t denote the exogenous state of the economy in period t. The state has continuous support and may be multidimensional. The economy is populated by a finite number of infinitely lived agents, $h \in \mathbb{H} = \{1 \ldots H\}$. Agents choose individual consumption at time t as a function of the entire history of the exogenous state, y^t, where $y^t = (y_0, \ldots, y_t)$. Let $C^h(y^t)$ be the individual consumption for agent h. Similarly, $C(y^t) \in \mathbb{R}_{++}$ denotes the aggregate consumption of all agents as a function of the history, y^t. The individual consumption levels satisfy the usual market-clearing condition,

$$\sum_{h=1}^{H} C^h(y^t) = C(y^t). \tag{1}$$

Agents have subjective beliefs about the stochastic process of the exogenous state. We denote the expectation operator for agent h at time t by E_t^h. Each agent has recursive utility. Let $\{C^h\}_t = \{C^h(y^t), C^h(y^{t+1}), \ldots\}$ denote the consumption stream of agent h from time t forward. The utility of agent h at time t, $U^h(\{C^h\}_t)$, is specified by an aggregator, $F^h(c, x)$, and a certainty–equivalence, $G^h(x)$,

$$U^h(\{C^h\}_t) = F^h\left(C^h(y^t), R_t^h\left[U^h(\{C^h\}_{t+1})\right]\right), \tag{2}$$

with

$$R_t^h[x] = G_h^{-1}\left(E_t^h[G_h(x)]\right). \tag{3}$$

We assume that the functions F^h and G^h are both continuously differentiable. This preference framework includes both Epstein–Zin utility and discounted expected utility, for the appropriate choices of F^h and G^h. To simplify the analysis, we ensure that agents never choose zero consumption, in any state of the world, by imposing an Inada condition on the aggregator F^h; so, $F_1^h(c,x) \to \infty$ as $c \to 0$, where F_1^h denotes the derivative of F^h with respect to the first argument.

We also impose a condition on the agents' beliefs. Let $P_{t,t+1}^h$ be the subjective conditional distribution of y_{t+1} given y^t, and $P_{t,t+1}$ be the true conditional distribution. We assume that each agent's expectation can be written in terms of the true distribution as

$$E_t^h[x] = E_t\left[x \frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}\right],$$

for some measurable function $\mathrm{dP}_{t,t+1}^h/\mathrm{dP}_{t,t+1}$. In mathematical terms, every agent's conditional distribution is absolutely continuous with respect to the true distribution. Then, by the Radon–Nikodym theorem, see Billingsley (1999, Chapter 32), such a $\mathrm{dP}_{t,t+1}^h/\mathrm{dP}_{t,t+1}$ must exist. Accordingly, $\mathrm{dP}_{t,t+1}^h/\mathrm{dP}_{t,t+1}$ is known as the Radon–Nikodym derivative of $\mathrm{P}_{t,t+1}^h$ with respect to $\mathrm{P}_{t,t+1}$. We also assume that, vice versa, the true distribution is absolutely continuous with respect to every agent's subjective distribution.

To solve for equilibrium, we assume that markets are complete so that we can reformulate equilibrium as a social welfare problem (Mas-Colell and Zame (1991)). The social planner maximizes a weighted sum of the individual agents' utilities at $t = 0$. Let $\boldsymbol{\lambda} = \left(\bar{\lambda}^1, \ldots, \bar{\lambda}^H\right) \in \mathbb{R}_{++}^H$ be a vector of positive Negishi weights and let $\{\boldsymbol{C}\}_0 = \left(\{C^1\}_0, \ldots, \{C^H\}_0\right)$ be an H-vector of the agents' consumption processes. Then, the social planner maximizes

$$SP(\{\boldsymbol{C}\}_0; \boldsymbol{\lambda}) = \sum_{h=1}^{H} \bar{\lambda}^h U^h\left(\{C^h\}_0\right) \tag{4}$$

subject to the market-clearing Equation (1). We denote an optimal solution to the social planner's problem for given Negishi weights $\boldsymbol{\lambda}$ by $\{\boldsymbol{C}\}_0^*$. For each agent $h \in \mathbb{H}$, let $U_t^h = U^h(\{C^h\}_t^*)$ be the utility in period t at the optimal solution. Also, for ease of notation, we suppress the state dependence of consumption and simply write C_t^h for $C^h(y^t)$.

Theorem 1. *The vector of consumption processes $\{\boldsymbol{C}\}_0^*$ solves the social planner's problem (4,1) for given Negishi weights $\boldsymbol{\lambda} = \left(\bar{\lambda}^1, \ldots, \bar{\lambda}^H\right)$ if and only if the consumption processes* *satisfy the following first-order conditions in each period $t \geq 0$;*

$$\lambda_t^h F_1^h(C_t^h, R_t^h\left[U_{t+1}^h\right]) = \lambda_t^1 F_1^1(C_t^1, R_t^1\left[U_{t+1}^1\right]), \tag{5}$$

where the weights λ_t^h satisfy

$$\lambda_0^h = \bar{\lambda}^h, \tag{6}$$

$$\frac{\lambda_{t+1}^h}{\lambda_{t+1}^1} = \frac{\Pi_{t+1}^h}{\Pi_{t+1}^1}\frac{\lambda_t^h}{\lambda_t^1}, \qquad t \geq 0, h \in \{2, \ldots H\}, \tag{7}$$

with Π_{t+1}^h given by

$$\Pi_{t+1}^h = F_2^h\left(C_t^h, R_t^h[U_{t+1}^h]\right) \cdot \frac{G_h'(U_{t+1}^h)}{G_h'(R_t^h[U_{t+1}^h])}\frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}. \tag{8}$$

Appendix A contains the proof of this theorem as well as those of the theoretical results presented later in this section.

In each period t, the weights λ_t^h are only determined up to a scalar factor, so we are free to choose a normalization. For numerical purposes, the normalization requiring the weights λ_t^h to lie in the unit simplex in every period is convenient. From a conceptual point of view, an attractive choice is to let $\lambda_{t+1}^1 = \Pi_{t+1}^1 \lambda_t^1$, because then for all h, $\lambda_{t+1}^h = \Pi_{t+1}^h \lambda_t^h$.

If the aggregator F^h is additively separable, then the allocation of consumption in (5) depends only on the current value for the weights λ_t^h. Additive separability is the most common case in applications. Discounted expected utility is additively separable, while Epstein–Zin can be transformed to be so. In this particular case, the Negishi weights and individual agents' consumption allocations are closely linked. The following theorem provides an asymptotic result relating the limits of weights λ_t^h to the limits of consumption.

Theorem 2. *Suppose that F^h is additively separable for all $h \in \mathbb{H}$ and that the aggregate endowment is bounded, $C_t \in [\underline{C}, \overline{C}]$ for finite constants $\overline{C} \geq \underline{C} > 0$. If $\lambda_t^j/\lambda_t^i \to \infty$, then $C_t^i \to 0$. If $C_t^i \to 0$, then for at least one other agent j, $\limsup_t \lambda_t^j/\lambda_t^i = \infty$.*

Note that $\limsup_t \lambda_j/\lambda_i$ is a random variable—the limit can depend on the history. Theorem 2 generalizes a similar result by Blume and Easley (2006).

2.2 The Growth Economy with Epstein–Zin Preferences

We now consider the special case of our heterogeneous-agent economy in which aggregate consumption is expressed exogenously in terms of growth rates and agents have Epstein–Zin preferences (see Epstein and Zin (1989) and Weil (1989)). For this popular parametrization of asset-pricing models, we can sharpen the general results of Theorems 1 and 2. Here we state the equilibrium conditions for this model parametrization and refer any interested reader to Appendix A.2 for a proper derivation of those conditions.

If agent h has Epstein–Zin preferences, then

$$F^h(c,x) = \left[(1-\delta^h)c^{\rho^h} + \delta^h x^{\rho^h}\right]^{1/\rho^h} \tag{9}$$

$$G_h(x) = x^{\alpha^h} \tag{10}$$

with parameters $\rho^h \neq 0, \alpha^h < 1$. In this case, the equations are all homogeneous, so we can divide through by aggregate consumption and express the equilibrium allocations in terms of individual consumption shares, $s_t^h = C_t^h/C_t$. Market-clearing (1) implies that

$$\sum_{h=1}^{H} s_t^h = 1. \tag{11}$$

Let V_t^h be agent h's value function. We also normalize this function by aggregate consumption, $v_t^h = V_t^h/C_t$. Let $c_t = \log C_t$ and $\Delta c_{t+1} = c_{t+1} - c_t$. The normalized value function of agent h satisfies the following fixed-point equation,

$$v_t^h = \left[(1-\delta^h)(s_t^h)^{\rho^h} + \delta^h R_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\rho^h}\right]^{\frac{1}{\rho^h}}, \qquad h \in \mathbb{H}, \tag{12}$$

with $R_t^h(x) = \left(E_t^h\left[x^{\alpha^h}\right]\right)^{\frac{1}{\alpha^h}}$. The parameter δ^h is the discount factor, $\rho^h = 1 - \frac{1}{\psi^h}$ determines the elasticity of intertemporal substitution (EIS), ψ^h, and $\alpha^h = 1-\gamma^h$ determines the relative risk aversion, γ^h, of agent h.

To accompany the normalized value function we introduce a normalized Negishi weight, $\underline{\lambda}_t^h = \frac{\lambda_t^h}{(v_t^h)^{\rho^h - 1}}$. In Appendix A.2 we show that the consumption share s_t^h of agent h is given by

$$\underline{\lambda}_t^h(1-\delta^h)(s_t^h)^{\rho^h-1} = \underline{\lambda}_t^1(1-\delta^1)(s_t^1)^{\rho^1-1}. \tag{13}$$

Finally, the equations for $\underline{\lambda}_t^h$ simplify to

$$\begin{aligned}
\frac{\underline{\lambda}_{t+1}^h}{\underline{\lambda}_{t+1}^1} &= \frac{\underline{\Pi}_{t+1}^h}{\underline{\Pi}_{t+1}^1}\frac{\underline{\lambda}_{t+1}^h}{\underline{\lambda}_{t+1}^1} \\
\underline{\Pi}_{t+1}^h &= \delta^h e^{\rho^h \Delta c_{t+1}} \frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}} \frac{\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h - \rho^h}}{R_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}, \qquad h \in \mathbb{H}^-.
\end{aligned}$$

This simplification gives us $H-1$ nonlinear equations for the equilibrium. In our numerical calculation, we complete the system by requiring that $\sum \underline{\lambda}_t^h = 1$, when we solve for the weights, $\underline{\lambda}_t^h$, given by

$$\begin{aligned} \underline{\lambda}_{t+1}^h &= \frac{\underline{\lambda}_t^h \underline{\Pi}_{t+1}^h}{\sum_{h=1}^H \underline{\lambda}_t^h \underline{\Pi}_{t+1}^h} \\ \underline{\Pi}_{t+1}^h &= \underbrace{\delta^h e^{\rho^h \Delta c_{t+1}} \frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}}_{\text{CRRA-Term}} \underbrace{\frac{\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h - \rho^h}}{R_t^h \left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h - \rho^h}}}_{\text{Additional EZ-Term}}, \qquad h \in \mathbb{H}^-. \end{aligned} \tag{14}$$

Unlike in the discounted expected utility case, the dynamics of the weights $\underline{\lambda}_t^h$ depend on the value functions (12), which in turn depend on the consumption decisions (13). Hence, to compute the equilibrium we need to jointly solve equations (11)–(14). As there are—to the best of our knowledge—no closed-form solutions for the general model, we present in Appendix B.1 a numerical solution approach, which is based on projection methods to approximate for the equilibrium functions.

In this setting, we can derive an improvement over Theorem 2—the limiting behavior for λ_t^h drives the limiting behavior for an agent's share of aggregate consumption. This result requires no assumptions on aggregate consumption, only that agents have utility in the Epstein–Zin family.

Theorem 3. *Suppose all agents in the economy have Epstein–Zin preferences. If $\underline{\lambda}_t^j / \underline{\lambda}_t^i \to \infty$, then $s_t^i \to 0$. If $s_t^i \to 0$, then for at least one agent j, $\limsup_t \underline{\lambda}_t^j / \underline{\lambda}_t^i = \infty$.*

This completes our discussion of the theoretical framework for our analysis. Appendix A provides proofs for the three theorems in this section. Along the way, we derive a system of first-order conditions for Epstein–Zin preferences. This system constitutes the foundation for our numerical solution method (see Appendix B).

3 A Long-Run Risk Model with Differences in Beliefs

We consider a standard long-run risk model as in Bansal and Yaron (2004), in which log aggregate consumption growth Δc_{t+1} and log aggregate dividend growth Δd_{t+1} are given by

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + x_t + \sigma \eta_{c,t+1} \\ x_{t+1} &= \rho_x x_t + \phi_x \sigma \eta_{x,t+1} \\ \Delta d_{t+1} &= \mu_d + \Phi x_t + \phi_d \sigma \eta_{d,t+1} + \phi_{d,c} \sigma \eta_{c,t+1}. \end{aligned} \tag{15}$$

The process x_t captures the long-run variation in the mean of consumption and dividend growth and $\eta_{c,t+1}$, $\eta_{x,t+1}$, and $\eta_{d,t+1}$ are i.i.d. normal shocks. A key feature of long-run risk models is highly persistent shifts in the growth rate of consumption. With a preference for the early resolution of risks ($\gamma > \frac{1}{\psi}$), investors will dislike shocks in x_t and require a large premium for bearing those risks. Hence, the results in the long-run risk literature rely on a highly persistent state process x_t, or, put differently, the parameter ρ_x needs to be very close to 1 (0.979 in the original calibration of Bansal and Yaron (2004)).

In this paper, we analyze the equilibrium implications of differences in beliefs with regard to the long-run risk process. As x_t is not directly observable from the data, it is reasonable to assume that investors disagree—at least slightly—about the data generating process of x_t. In light of the results from the representative-agent literature on long-run risks, the majority of investors need to believe in a highly persistent long-run risk process. Otherwise, asset prices would be determined by those investors who don't believe, or who believe less, in long-run risks; and, hence, the model outcomes would certainly not be consistent with the data. Therefore, we assume that a majority of investors believe in a highly persistent long-run risk process. Then we address the question of what happens if there is a small fraction of investors who believe in slightly less persistent shocks—that is, who are somewhat skeptical of the presence of long-run risks.

3.1 The Benchmark Economy

Our baseline setup is an economy with $H = 2$ agents in which the first agent believes that ρ_x is close to 1 while the second agent believes that ρ_x is slightly smaller. We do not make a specific assumption about which agent has the correct beliefs. In fact, we show below that for small belief differences the true distribution has a negligible influence on equilibrium outcomes. We denote by ρ_x^h the belief of agent h about ρ_x. As x_{t+1} conditional on time t information is normally distributed with mean $\rho_x x_t$ and variance $\phi_x^2 \sigma^2$, agents' beliefs $\mathrm{dP}^h_{t,t+1}$ are given by

$$\mathrm{dP}^h_{t,t+1} = \frac{1}{\sqrt{2\pi}\phi_x\sigma} \exp\left(-\frac{1}{2}\left(\frac{x_{t+1} - \rho_x^h x_t}{\phi_x \sigma}\right)^2\right) dx_{t+1}.$$

We can think of this model as an extension of Borovička (2015), who considers a two-agent setup with different beliefs about the mean growth rate of the economy. For Epstein–Zin preferences, Borovička (2015) shows that the agent with the more optimistic beliefs (a larger belief about the mean growth rate) will dominate the economy in the long run as long as the risk aversion in the economy is sufficiently large. This result stands in stark contrast to the case of CRRA preferences, where the agent with the more correct beliefs will always dominate, independent of the choice of preference parameters (see, for example, Yan (2008)).[3]

In our model with different beliefs about the persistence of long-run risks, agents' beliefs about the mean growth rate change over time relative to one another. The time-t expectation of agent h about the mean growth rate is given by $\rho_x^h x_t$. In the benchmark case with $\rho_x = \rho_x^1 > \rho_x^2$, a negative realization of x_t implies $\rho_x^2 x_t > \rho_x^1 x_t$ and, therefore, the second agent is more optimistic (less pessimistic). For $x_t > 0$, we have $\rho_x^2 x_t < \rho_x^1 x_t$; now the first agent is more optimistic about the mean growth rate. Thus, we can think of this model as a time-varying version of Borovička (2015), in which the beliefs about the growth rate change over time.

Most long-run risk models calibrate the underlying cash-flow parameters in order to match asset-pricing data. For example, Bansal and Yaron (2004) use a value of $\rho_x = 0.979$. Bansal, Kiku, and Yaron (2012) use $\rho_x = 0.975$, and Drechsler and Yaron (2011) assume $\rho_x = 0.976$. They obtain high values of ρ_x by construction, as otherwise the models would not be consistent with the high equity premium observed in the data. The study by Bansal, Kiku, and Yaron (2016) relies on cash flow and asset-pricing data to estimate the long-run risk model parameters and reports a value of $\rho_x \approx 0.98$ with a standard error of 0.01. For our baseline calibration, we assume that the first agent believes that $\rho_x^1 = 0.985$. This value implies an equity premium of 6.53% for the representative-agent economy, which is consistent with the value observed in the data. The second agent has slightly smaller beliefs about the persistence, with $\rho_x^2 = 0.975$. Both values lie well within the confidence interval provided by Bansal, Kiku, and Yaron (2016). A small change in ρ_x has large effects on asset prices. For $\rho_x = 0.975$, the equity premium decreases to 2.76% in the representative-agent economy. For $\rho_x = 0.95$, it collapses to 0.26% and the influence of x_t on asset prices is negligible. For completion, we also analyze the model for the values $\rho_x^1 = 0.985$ and $\rho_x^2 = 0.95$ of the persistence parameter.

While the two agents have different beliefs, they have identical Epstein–Zin utility parameters in the benchmark economy (we relax this assumption in Section 5). They share the properties of the representative agent of Bansal and Yaron (2004) with $\psi^1 = \psi^2 = 1.5$, $\gamma^1 = \gamma^2 = 10$, and $\delta^1 = \delta^2 = 0.998$. For the remaining parameters of the state processes (15) we also use the calibration from Bansal and Yaron (2004), with $\mu_c = \mu_d = 0.0015, \sigma = 0.0078, \Phi = 3, \phi_d = 4.5, \phi_{d,c} = 0$, and $\phi_x = 0.044$. (This calibration is used for all results in the present paper.)

[3] In our benchmark economy, the only difference between the agents is their beliefs about the state processes; they share the same utility parameter specifications. In Section 5 we present results for the model with heterogeneous preferences.

3.2 Plausibility of Persistent Belief Differences

The benchmark economy exhibits two agents with persistent belief differences. A critical reader may argue that this assumption is unrealistic, since we may expect the agents to learn the true exogenous growth processes over time. To address such potential criticism, we now examine the speed of learning in the long-run risk model. For this purpose, we suppose that investors need to estimate model parameters from the data. We show that it is difficult to obtain a precise estimate for the persistence parameter ρ_x of the x_t process in small finite samples. Very long time series—much longer than those observed in our simulations (of up to 500 years)—are required for the belief differences to vanish. Therefore, learning the true persistence parameter is a very slow process.

Suppose the true persistence parameter of the long-run risk process is $\rho_x = 0.985$, which is just the value of ρ_x^1 in the benchmark economy. Now suppose an investor does not know this parameter but estimates it from a finite sample. To analyze this estimation, we simulate 1'000 time series consisting of 500 years of monthly data and calculate estimates after 100, 200, and 500 years. As a first estimation approach, we assume that the investor directly observes x_t and simply estimates the AR(1) process

$$x_{t+1} = \mu_x + \rho_x x_t + \sigma_x \eta_{x,t+1}. \tag{16}$$

We distinguish two cases of this estimation approach; first, the investor estimates the process with the constant μ_x; and second, the investor estimates the process without a constant and knows that $\mu_x = 0$. We use least-squares to obtain consistent estimates. In reality, the process x_t is not directly observable but must be inferred from the consumption growth time series. Therefore, as a second approach, we also estimate the full state-space model (15) using the Kalman filter:

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + x_t + \sigma \eta_{c,t+1} \\ x_{t+1} &= \rho_x x_t + \sigma_x \eta_{x,t+1}. \end{aligned} \tag{17}$$

Table 1 reports the results of the two estimation approaches. We observe that for 100 years of data there is the usual significant finite-sample downward bias in the mean of the point estimates $\hat{\rho}_x$ (see, for example, James and Smith (1998)). Kendall (1954) shows that the bias is approximately $-(1+3\rho_x)/T = -0.0033$ for the model with a constant, which is in accordance with the value we observe. (The investor can approximate the bias using the point estimate $\hat{\rho}_x$ and the number of periods, $T = 1200$.) The table also reports the 5% and 1% quantiles of the point estimates from the 1'000 simulations. After 100 years, even after adjusting for

Table 1: Parameter Estimates from Simulated Data

		x_t observable		x_t unobservable
		with constant	w/o constant	all parameters
	$\hat{\rho}_x$	0.9817	0.9837	0.9715
100 Years	$\hat{\rho}_{x,0.05}$	0.9709	0.9742	0.9329
	$\hat{\rho}_{x,0.01}$	0.9660	0.9692	0.8252
	$\hat{\rho}_x$	0.9834	0.9844	0.9806
200 Years	$\hat{\rho}_{x,0.05}$	0.9760	0.9773	0.9627
	$\hat{\rho}_{x,0.01}$	0.9727	0.9742	0.9408
	$\hat{\rho}_x$	0.9844	0.9847	0.9833
500 Years	$\hat{\rho}_{x,0.05}$	0.9804	0.9810	0.9745
	$\hat{\rho}_{x,0.01}$	0.9786	0.9789	0.9691

The table shows the mean point estimates of ρ_x as well as the 5% and 1% quantiles after 100, 200, and 500 years obtained from simulating 1'000 monthly time series of data. In the first approach, Equation (16) is used for x_t, assuming the process is directly observable, and least-squares is used to estimate the model parameters; we distinguish the two cases of estimating the AR(1) model with and without a constant. The second approach assumes that x_t is unobservable and the full state-space model (17) is estimated using the Kalman filter. For the data generating process, we use $\mu_x = 0, \rho_x = 0.985, \sigma_x = 0.0003432, \mu = 0.0015$, and $\sigma = 0.0078$.

the bias, the 5% quantile is still smaller than $\rho_x^2 = 0.975$ in the benchmark economy. After 200 years, again after adjusting for the bias, the 1% quantile is still smaller than 0.975. If the investor knows that $\mu_x = 0$, then both the standard errors of the estimation and the bias become slightly smaller.

In reality, however, the investor does not observe x_t but must estimate the full model (17). In this case, the bias in $\hat{\rho}_x$ becomes significantly larger with a mean value of 0.9715 and a 5% quantile value of 0.9329. Hence, also the second value of $\rho_x^2 = 0.95$ used for the second agent is well above this quantile after 100 years. After 200 years, the standard error and the bias become smaller, but a value of 0.95 is still well within the 1% quantile. After 500 years the bias slowly vanishes but $\rho_x^2 = 0.975$ is still within the 1% range (even after correcting for a bias).

In light of the estimation results, we conclude that even if the investor might learn about the true data generating process after 500 or more years, it is reasonable to assume that any nontrivial initial belief differences persist for at least 100 years if not for much longer.

3.3 Additional Model Specifications

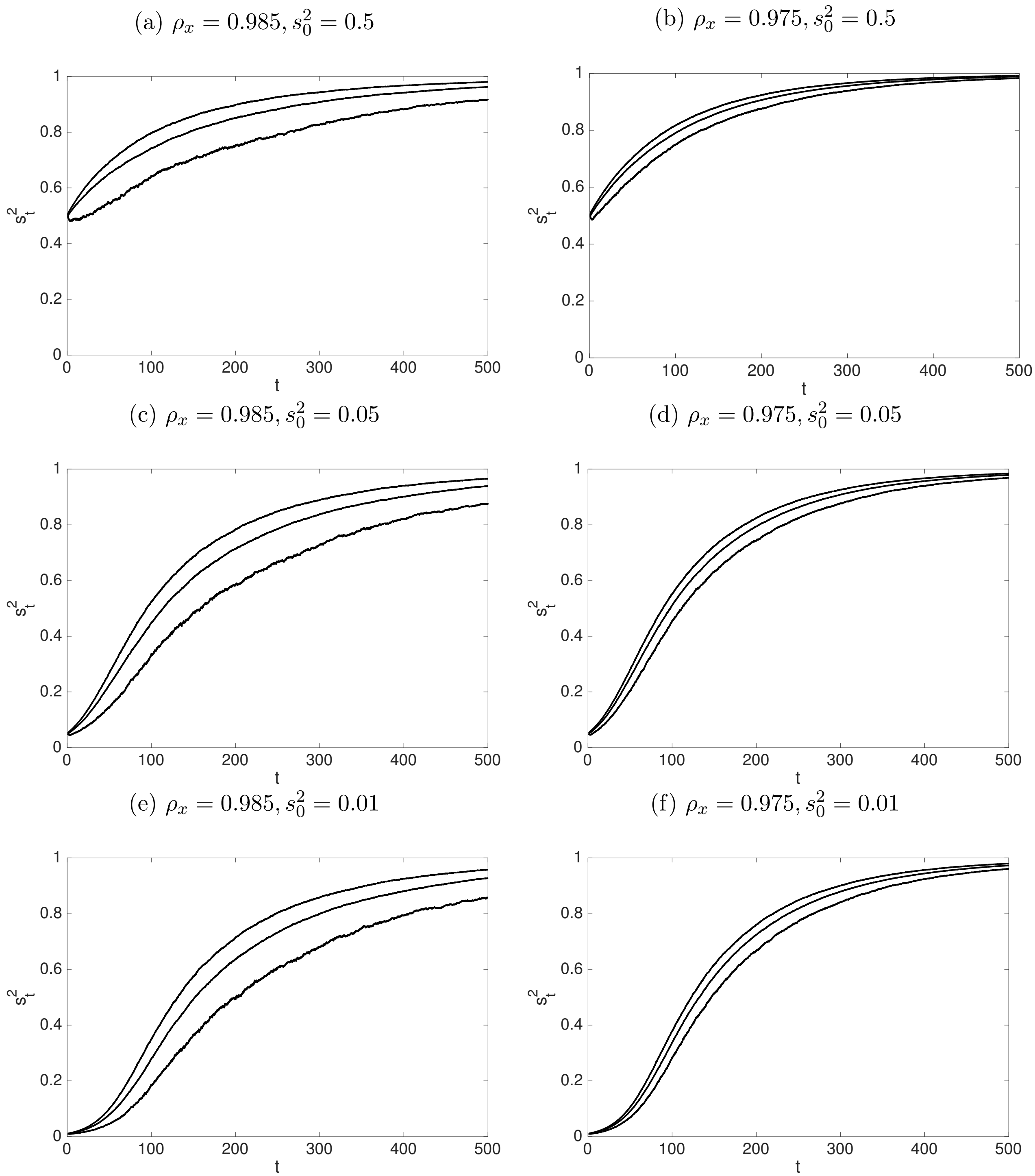
The main focus of the present paper is a thorough analysis of the benchmark economy of two agents with identical preferences and heterogeneous beliefs about the persistence of the long-run risk process. However, Yan (2008) argues that for CRRA preferences and i.i.d. consumption growth, a difference in the beliefs can be offset by only a slight variation in the preference parameters. In Section 5 we, therefore, consider the case in which investors have also different preference parameters. First, we analyze the model in which the two agents both know the true persistence parameter, $\rho_x = 0.985$, but differ in their EZ utility parameters. We report results for two cases. In the first case, agents have the same EIS but different risk aversion parameters. In the second case, agents have identical risk aversion parameters but differ in their EIS. We find that for differences in the EIS, the changes in the consumption shares are rather slow, while they move faster for different degrees of risk aversion. Therefore, we then search for risk aversion parameters that lead to consumption shares with a roughly constant median in the economy with different beliefs. We find that—in contrast to the findings of Yan (2008) for CRRA preferences—a large difference in the risk aversion is required to offset the trend in the consumption shares stemming from the differences in beliefs.

4 Heterogeneous Beliefs about Persistence

We begin with the analysis of the equilibrium dynamics of the consumption shares of the individual agents. Figure 1 shows the consumption share of the second, skeptical agent ($\rho_x^2 = 0.975$) over time for different initial shares $s_0^2 = \{0.01, 0.05, 0.5\}$. We report the median, 5%, and 95% quantile paths using 1'000 samples each consisting of 500 years of simulated data. To minimize the influence of the initial value of x_t, we initialize each simulated path by running a "burn-in" period of 1'000 years before using the output. The left panel shows the results for $\rho_x = \rho_x^1 = 0.985$ (the first agent has correct beliefs) and the right panel for $\rho_x = \rho_x^2 = 0.975$ (the second agent has correct beliefs).

We observe that in all cases the consumption share of the skeptical agent 2 strongly increases over time. While this increase occurs faster if agent 2 has the correct beliefs (right panel) the increase is almost as strong if agent 1 has the correct beliefs (left panel). Hence, given a small difference in the beliefs, independent of whether agent 1 or agent 2 has the correct beliefs, in the long run the agent with the lower beliefs about ρ_x will dominate the economy. Most importantly, even if the economy is initially almost entirely populated by agent

Figure 1: Consumption Shares—Simulations



(a) $\rho_x = 0.985, s_0^2 = 0.5$

(b) $\rho_x = 0.975, s_0^2 = 0.5$

(c) $\rho_x = 0.985, s_0^2 = 0.05$

(d) $\rho_x = 0.975, s_0^2 = 0.05$

(e) $\rho_x = 0.985, s_0^2 = 0.01$

(f) $\rho_x = 0.975, s_0^2 = 0.01$

The figure shows the median, 5%, and 95% quantile paths of the consumption share of agent 2 for 1'000 samples each consisting of 500 years of simulated data. Agent 2 believes that $\rho_x^2 = 0.975$ and agent 1 believes that $\rho_x^1 = 0.985$. Results are shown for different initial consumption shares ($s_0^2 = \{0.01, 0.05, 0.5\}$). The left panel depicts the case where the skeptical agent, agent 2, has the wrong beliefs about the long-run risk process ($\rho_x = 0.985 = \rho_x^1$) and in the right panel the skeptical agent has the right beliefs ($\rho_x = 0.95 = \rho_x^2$).

1 ($s_0^2 = 0.01$), his consumption share decreases sharply and he loses significant share in a short amount of time. Table 2 reports the corresponding median consumption shares for different time horizons for $s_0^2 = \{0.01, 0.05, 0.5\}$. We observe that for $s_0^2 = 0.01$ the consumption share of agent 1 has decreased by more than 27% after 100 years, 62% after 200 years, and almost 92% after 500 years.

Table 2: Consumption Shares—Summary Statistics

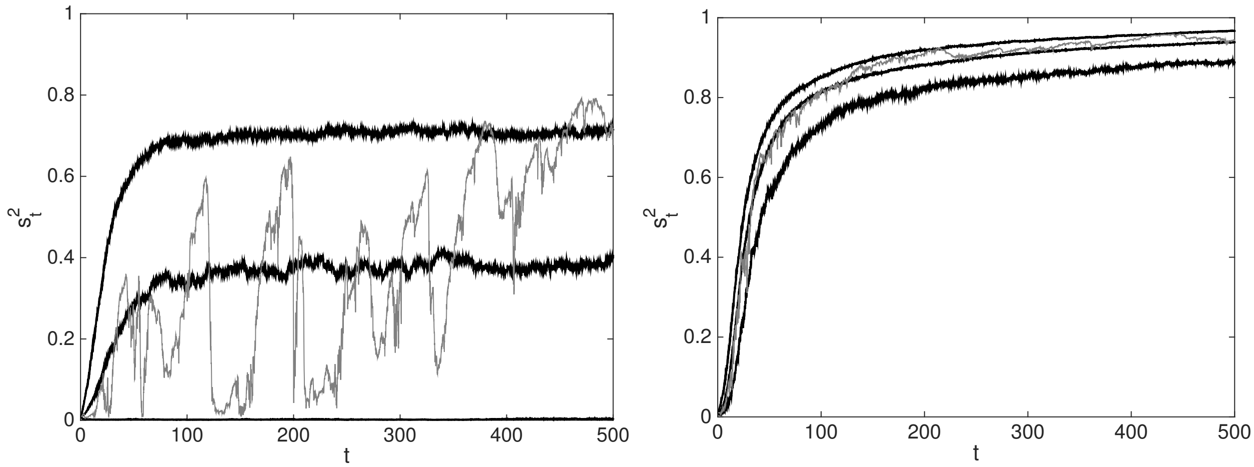
	$\rho_x = 0.985$			$\rho_x = 0.975$		
Years	100	200	500	100	200	500
$s_0^2 = 0.5$	0.7429 (0.0500)	0.8515 (0.0481)	0.9628 (0.0212)	0.8904 (0.0118)	0.9683 (0.0072)	0.9995 (4.2e-5)
$s_0^2 = 0.05$	0.4507 (0.0589)	0.7143 (0.0636)	0.9393 (0.0293)	0.5140 (0.0301)	0.7947 (0.0252)	0.9787 (0.0049)
$s_0^2 = 0.01$	0.2824 (0.0509)	0.6376 (0.0681)	0.9278 (0.0326)	0.3404 (0.0294)	0.7249 (0.0293)	0.9732 (0.0060)

The table shows the median and the standard deviation (in parenthesis) of the consumption share of agent 2 using 1'000 samples each consisting of 500 years of simulated data. Agent 2 believes that $\rho_x^2 = 0.975$ and agent 1 believes that $\rho_x^1 = 0.985$. Summary statistics are shown for different initial consumption shares ($s_0^2 = \{0.01, 0.05, 0.5\}$) and different time periods $T = \{100, 200, 500\}$ years. The left panel depicts the case where the skeptical agent, agent 2, has the wrong beliefs about the long-run risk process ($\rho_x = 0.985 = \rho_x^1$) and in the right panel the skeptical agent has the right beliefs ($\rho_x = 0.95 = \rho_x^2$).

Figure 2 shows the corresponding results for $\rho_x^2 = 0.95$ and an initial allocation of $s_0^2 = 0.01$. The left panel shows the results for $\rho_x = 0.985$ (agent 1 has the correct beliefs). We observe that the initial increase in the consumption share is stronger compared to the case with $\rho_x^2 = 0.975$ but that the median share does not become as large in the long run (the median shares of the second agent after 100, 200, and 500 years are given by 32.59%, 37.82%, and 40.19%, respectively). Also, the 5% and 95% quantile paths show that there is significantly more variation in the shares. The figure also shows a sample path (grey line). We observe that there are large drops and recoveries in the consumption share. The large drops occur, because the second agent assigns "wrong" probabilities to extreme states and hence "bets" on states that turn out to occur less often in the long run. This effect works in favor of agent 2, once she has the correct beliefs and is therefore more likely to bet on the correct states. This case is shown in the right panel ($\rho_x = \rho_x^2 = 0.95$), where we indeed observe that the increase in the consumption share is much stronger and that the large drops in consumption are no longer present. The recoveries in the left panel occur because the second agent is less

Figure 2: Consumption Shares for $\rho_x^2 = 0.95$—Simulations

(a) $\rho_x = 0.985$ (b) $\rho_x = 0.95$



The figure shows the median, 5%, and 95% quantile paths of the consumption share of agent 2 for 1'000 samples each consisting of 500 years of simulated data as well as a sample path (grey line). Agent 2 believes that $\rho_x^2 = 0.95$ and agent 1 believes that $\rho_x^1 = 0.985$. Results are shown for an initial consumption share of $s_0^2 = 0.01$. The left panel depicts the case where the skeptical agent, agent 2, has the wrong beliefs about the long-run risk process ($\rho_x = 0.985 = \rho_x^1$) and in the right panel the skeptical agent has the right beliefs ($\rho_x = 0.95 = \rho_x^2$).

afraid of long-run risks and hence sells insurance against these risks to the first agent. As the first agent believes that $\rho_x^1 = 0.985$, he strongly dislikes shocks in x_t and is willing to pay a high premium to insure against these risks. So there are two interacting effects that affect equilibrium outcomes. We provide a detailed analysis of the two effects later, in Section 4.1.

What does the change in the consumption shares imply for asset prices and aggregate financial market statistics? We assume that the economy is initially almost entirely populated by agent 1 in order to generate a high equity premium consistent with the data. But the consumption share of the first agent decreases rapidly, and so will that agent's influence on asset prices. In Table 3 we show the annualized equity premium in the years 0, 100, 200, and 500, assuming an initial share of $s_0^2 = 0.01$.[4] The left panel shows the results for $\rho_x^2 = 0.975$ where agent 1 has the correct beliefs. For the initial allocation $s_t^2 = 0.01$, when agent 1 dominates the economy, the aggregate risk premium is 6.42%. This value is very close to that of the representative-agent economy populated only by the first agent which generates a premium of 6.53%. After 100 years, when the share of agent 1 has decreased from 99% to 72%,

[4]Note that Table 3 does not report the premium starting with a given value for s_0^2 and simulating a long time series, but that we report the average premium for a given consumption share $s_t^2 = \bar{s}$. Hence, we take the expectation over all x_t while keeping the consumption share constant at $\bar{s}$. The population moment for 500 years of simulated data is given in Table 4.

Table 3: Equity Premium for Different Consumption Shares

	$\boldsymbol{\rho_x^2 = 0.975}$		$\boldsymbol{\rho_x^2 = 0.95}$	
	s_t^2	Equity Premium	s_t^2	Equity Premium
Rep. Agent 1	0	6.53	0	6.53
0 Years	0.01	6.42	0.01	5.42
100 Years	0.2824	4.59	0.3259	1.84
200 Years	0.6376	3.49	0.3782	1.64
500 Years	0.9278	2.89	0.4019	1.56
Rep. Agent 2	1	2.76	1	0.26

The table shows the annualized equity premium for a specific consumption share $s_t^2 = \bar{s}$. The premium is reported for the equilibrium allocations after 0, 100, 200, and 500 years of simulated data assuming an initial share of $s_0^2 = 0.01$ (see Table 2). Agent 1 has the correct beliefs with $\rho_x^1 = \rho_x = 0.985$. The left panel depicts the case for $\rho_x^2 = 0.975$ and the right panel for $\rho_x^2 = 0.95$.

the premium decreases to 4.59%. Hence, even if agent 1 holds almost all wealth initially, which implies a high risk premium, the premium will drop by almost 2% within a century. After 200 years, the premium decreases by almost 3% and after 500 years it is almost at the level of the representative-agent economy populated only by agent 2, with a premium of 2.89%. The right panel shows the corresponding results for $\rho_x^2 = 0.95$. We observe that the sharp increase in the consumption share decreases the premium from 5.42% initially to 1.84% after 100 years—a decrease of more than 3.5% in a century. Hence, the difference in beliefs brings down the equity premium to well below the levels observed in the data even if the agent who is skeptical about the presence of long-run risks does not have the correct beliefs. (In Table 5 in Appendix C we show the corresponding results for the case where agent 2 rather than agent 1 has the correct beliefs. As expected, we observe that the drop in the equity premium is even more severe.)

Table 4 shows selected moments from the 1'000 sample paths starting with an initial share of $s_0^2 = 0.01$. We report the mean and the standard deviation of the annualized log price–dividend ratio, the annualized equity premium, and the risk-free return. Results are shown for the case in which agent 1 has the correct beliefs. In addition to the two-agent economy, the table also shows the two representative-agent cases where the economy is populated only by agent 1 ($s_t^2 = 0$) or agent 2 ($s_t^2 = 1$).

While the mean statistics of the two-agent economy lie well within the bands of the two representative-agent economies and depict the wealth shift towards the second agent, we observe that the volatility of the log price–dividend ratio is significantly larger for the two-

Table 4: Annualized Asset-Pricing Moments

	$E(p_t - d_t)$	$\sigma(p_t - d_t)$	$E\left(r_t^m - r_t^f\right)$	$E\left(r_t^f\right)$	$\sigma(r_t^m)$	$\sigma\left(r_t^f\right)$
			$\boldsymbol{\rho_x^2 = 0.975}$			
$s_t^2 = 0$	2.68	0.25	6.53	2.32	17.84	1.50
Two-Agent Economy	3.10	0.29	3.98	2.58	17.19	1.51
$s_t^2 = 1$	3.29	0.20	2.83	2.71	16.55	1.53
			$\boldsymbol{\rho_x^2 = 0.95}$			
$s_t^2 = 0$	2.68	0.25	6.53	2.32	17.84	1.50
Two-Agent Economy	3.60	0.48	2.63	2.47	20.37	1.58
$s_t^2 = 1$	6.27	0.14	0.26	2.93	14.80	1.52

The table shows selected moments from 1'000 samples each containing 500 years of simulated data starting with an initial share of $s_0^2 = 0.01$. It shows the mean and the standard deviation of the annualized log price–dividend ratio, the annualized market over the risk-free return, and the risk-free return. Agent 1 has the correct beliefs with $\rho_x^1 = \rho_x = 0.985$. All returns are shown in percent, so a value of 1.5 is a 1.5% annualized figure.

agent economy compared to both representative-agent economies. This effect is especially strong for $\rho_x^2 = 0.95$, where the volatility is 0.48 compared to 0.25 and 0.14 for the two representative-agent economies.

Both of these results are driven by shifts in the wealth distribution. In states of the world where one agent holds most of the wealth, the equity premium shifts towards the single-agent case for that agent. As the wealth distribution shifts between the agents, this averages the equity premium and other first-order moments between the two cases. For second-order moments, however, the time variation will drive them upward.

Beeler and Campbell (2012) argue that one of the major issues of the long-run risk models of Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012) is that they significantly underestimate the volatility of the price–dividend ratio (they report values of 0.18 compared to 0.45 observed in the financial market data). Our results show that differences in beliefs can potentially resolve this puzzle, since they lead to a significant increase in the volatility figures.

In order to show that it is really the heterogeneity of the investors that generates the excess volatility, we additionally consider the case where we fix the consumption shares of the investors at the long-run median of $s_t^2 = 0.40$ and these consumption shares remain constant throughout the simulations—that is, the volatility of the consumption share is 0 (compared to 0.22 for the case with variable consumption shares). In this scenario, we obtain a volatility of the price–dividend ratio of only 0.17, which is significantly smaller than the volatility of

0.48 for the case with variable consumption shares. This result suggests that the shifts of the consumption shares add significant excess volatility to the model. This strong increase in volatility can be explained by the large variation in the consumption shares for the case of $\rho_x^2 = 0.95$ (see Figure 2). Variation in the shares implies that the influence of each agent on asset prices varies over time. As both agents have significantly different price–dividend ratios in the representative-agent economies (a mean value of 2.68 for agent 1 compared to 6.27 for agent 2), the variation in the consumption shares generates excess volatility for the price–dividend ratio.

In sum, if there are different investors who all believe in long-run risks but use slightly different estimates for the long-run risk process, the investor who is more skeptical about ρ_x eventually dominates the economy. The investor who believes in a larger value of ρ_x rapidly loses wealth, no matter whether his beliefs are correct or not. Recall that a large ρ_x is needed to obtain a high risk premium in the long-run risk model. Even if this investor with the belief in a large ρ_x almost entirely populates the economy initially, that investor's consumption share decreases so fast that the equity premium in the economy declines considerably in a short amount of time. On the positive side, different beliefs about ρ_x introduce variations in the consumption shares, which in turn increase the volatility of the price–dividend ratio and generate a value closer to the level observed in the data. We have also seen that when the more skeptical agent's estimate of ρ_x of is a bit further away from the true value then both agents may survive in the long run. (And, obviously, the skeptical agent would not survive when his estimate is sufficiently small and thus sufficiently far away from the correct value of ρ_x.)

4.1 Optimal Consumption Decisions and Equilibrium Dynamics

In this section, we analyze the different effects that determine the equilibrium allocations of the agents. For this purpose, we discuss our results in relation to the findings of Borovička (2015). Borovička (2015) considers a simple two-agent economy with identical Epstein–Zin preferences and different beliefs about the mean growth rate of the economy.[5] Our model can be viewed as a generalized version of his model with time-varying beliefs about the mean growth rate in the economy. Borovička (2015) describes four channels through which individual choices influence long-run equilibrium dynamics: the *speculative bias channel*, the *risk premium channel*, the *savings channel*, and the *speculative volatility channel*. The *speculative volatility channel* only influences equilibrium outcomes for small degrees of risk aversion and has therefore a negligible

[5] In Borovička (2015) there is no long-run risk and log aggregate consumption growth is normally distributed.

influence on the results obtained in the present paper. In the following, we argue that the *speculative bias channel* and the *risk premium channel* can explain the equilibrium dynamics of the long-run risk model considered in the previous section, while the *savings channel* is rather irrelevant for our model specification.

4.1.1 The Speculative Bias Channel

The *speculative bias channel* alone determines equilibrium outcomes in the special case of CRRA preferences. The investors assign different subjective probabilities to future states and buy assets that pay off in states they believe are more likely. Hence, for CRRA utility the agent with the more correct beliefs will accumulate wealth in the long run, as the investor with the more distorted beliefs bets on states that have a vanishing probability under the true probability measure.

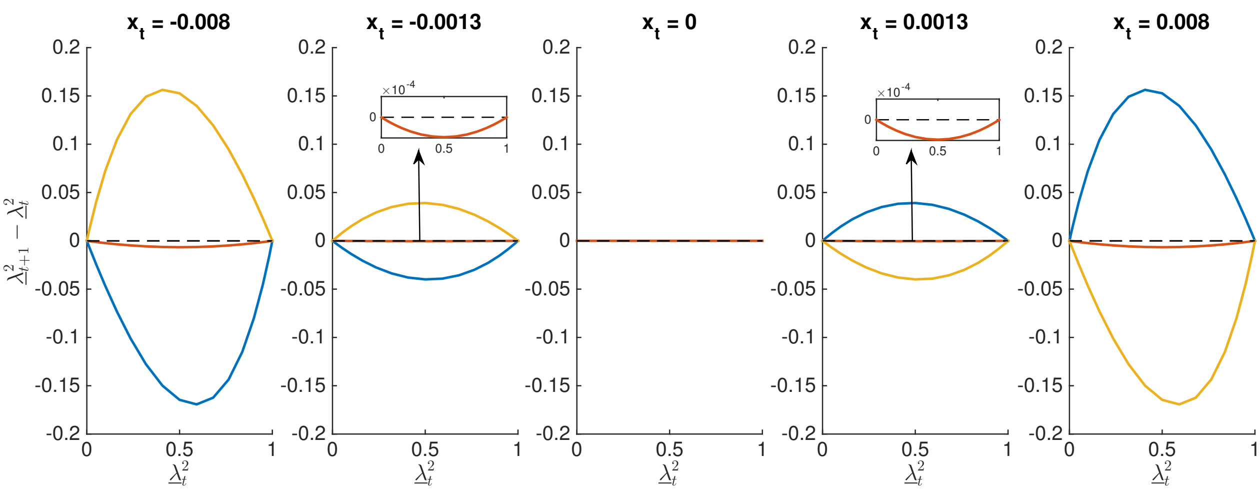
To demonstrate how the *speculative bias channel* affects equilibrium outcomes in the long-run risk model with different beliefs, we first consider the special case of CRRA preferences. In Figure 3 we show the change in the Pareto weights $\underline{\lambda}_{t+1}^2 - \underline{\lambda}_t^2$ as a function of $\underline{\lambda}_t^2$. Note that a positive (negative) change in the Pareto weight also implies a positive (negative) change in the consumption share (see Equation (13)). The blue and yellow lines depict the cases of a negative shock ($x_{t+1} - \rho_x x_t = -0.001$) and a positive shock ($x_{t+1} - \rho_x x_t = 0.001$) in x_{t+1}, respectively. The red line shows the average over all shocks. From left to right, the results are shown for $x_t = -0.008, x_t = -0.0013, x_t = 0, x_t = 0.0013$, and $x_t = 0.008$. Agent 1 has the correct belief, $\rho_x^1 = \rho_x = 0.985$, while agent 2 believes that $\rho_x^2 = 0.975$.

The second agent believes that x_t converges faster to its long-run mean than does agent 1. Hence, if $x_t < 0$, she assigns larger probabilities to large x_{t+1} and bets on those states as $\rho_x^2 x_t > \rho_x^1 x_t$ (left panels). The opposite holds true for $x_t > 0$. So agent 2 loses wealth if x_t is low and the shock in x_t is negative (blue line in the left-hand figures) or if x_t is high and the shock in x_t is also high (yellow line in the right-hand figures). Taking the average over all future realizations of x_{t+1} (red line), agent 2 loses wealth on average (red line). For $x_t = 0$ both agents share the same beliefs ($\rho_x^2 x_t = \rho_x^1 x_t$) and hence they assign the same probabilities to x_{t+1} (blue and yellow line coincide with the red line and are not visible). As agent 2 loses wealth on average for all x_t except for $x_t = 0$, she will eventually vanish in the long run. Note that the influence of the *speculative bias channel* becomes stronger the larger $|x_t|$ is, as the belief dispersion grows the more x_t deviates from its unconditional mean, $E(x_t) = 0$.

The *speculative bias channel* can be directly related to the two sets of results in the beginning of Section 4. There we show results for the case where agent 1 has the correct beliefs ($\rho_x = \rho_x^1$) as well as for the case where agent 2 has the correct beliefs ($\rho_x = \rho_x^2$). In the first case, the *speculative bias channel* works in favor of agent 1, while in the second case, it works in favor of agent 2. Hence, in case two, agent 2's consumption share increases more rapidly, as the *speculative bias channel* works in her favor (see Figure 1).

The *speculative bias channel* entirely determines the equilibrium in the standard case of CRRA preferences. For general Epstein–Zin preferences equilibrium dynamics become more complex. In the following we first describe the general effects of the *risk premium channel* and then analyze how the two effects interact and influence equilibrium outcomes.

Figure 3: Changes in the Wealth Distribution—The CRRA Case



The figure shows the change in the optimal weights $\underline{\lambda}^2_{t+1} - \underline{\lambda}^2_t$ as a function of $\underline{\lambda}^2_t$. From left to right, the change is shown for $x_t = \{-0.008, -0.0013, 0, 0.0013, 0.008\}$ ($\pm$ 4 standard deviations). The blue line depicts the case of a negative shock in x_{t+1} ($x_{t+1} - \rho_x x_t = -0.001$) and the yellow line of a positive shock in x_{t+1} ($x_{t+1} - \rho_x x_t = 0.001$). The red line shows the average over all shocks. Baseline calibration with $\rho_x = \rho_x^1$ and CRRA preferences.

4.1.2 The Risk Premium Channel

With Epstein–Zin preferences, risk-return trade-offs are not the same among agents and optimistic agents are willing to take larger risks (see Borovička (2015)). So if risk aversion, and hence risk premia, are high, more optimistic agents will profit from investing in a portfolio with a higher average return. Borovička (2015) calls this the *risk premium channel*. In our model, we cannot unambiguously specify optimists or pessimists, as beliefs about the mean growth rate change over time (see Section 3). We, rather, refer to agents who are skeptical about the presence of long-run risks, that is, they have a lower belief about ρ_x. Skepticism implies that an agent is less afraid of long-run risks. An investor who believes in a large ρ_x is afraid of large negative realizations of x_t and would therefore like to buy insurance against these risks. As risk premia in the economy are high due to the combination of high risk aversion, the preference for the early resolution of risks, and highly persistent shocks to x_t, the premium the investor is willing to pay will be high. The skeptical investor, on the other hand, will be willing to provide this insurance as she is less afraid of the long-run risks.

In Figure 4 we demonstrate how this channel affects model outcomes. It shows the corresponding results to Figure 3 but for the general case of Epstein–Zin preferences. First, consider the center panel, where $x_t = 0$ and hence the *speculative bias channel* has no effect on equilibrium outcomes (see Figure 3). Agent 1 is more afraid of negative shocks to x_{t+1} than is agent 2. Therefore, he buys insurance against the long-run risks, which pays off in bad times when there is a negative shock to x_{t+1} (the blue line is negative which implies an increase in the weights of the first agent for all $\underline{\lambda}_t^2$). Therefore, he has to pay a premium in good times. So, for a positive shock to x_{t+1} the results reverse (yellow line). The average over all shocks (red line) is positive, so agent 1 pays a positive premium to insure against long-run risk which is why he loses wealth on average. The effect is stronger for small $\underline{\lambda}_t^2$ and decreases for large $\underline{\lambda}_t^2$. A small value of $\underline{\lambda}_t^2$ implies that there is a large share of agents who want to buy insurance against long-run risks. Hence, they are willing to pay a higher price. The larger the share of the skeptical investors becomes, the lower becomes the demand for the insurance and, hence, the increase in the Pareto weights also becomes less pronounced.

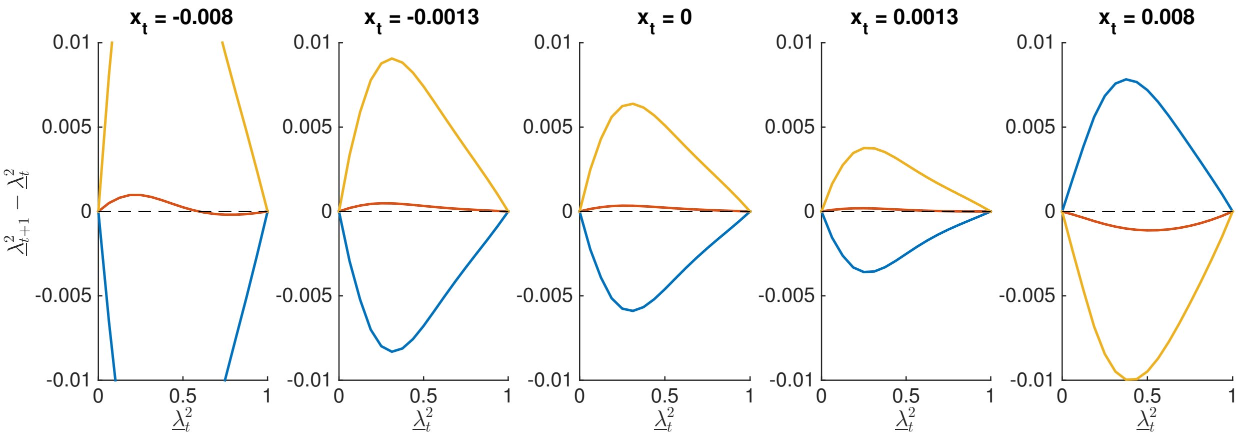
Decreasing x_t has two effects. First of all, agent 1 becomes more afraid of long-run risks (given a negative value of x_t, a large negative realization of x_{t+1} becomes more likely due to high persistence of ρ_x), which is why he wants to buy more insurance against long-run risks and is willing to pay a higher premium. We observe this effect in the second panel from the left ($x_t = -0.0013$) in Figure 4, where the average increase in the Pareto weight of the second agent (red line) increases compared to the results for $x_t = 0$. Additionally, the belief difference, and hence the difference between the subjective probabilities, becomes more pronounced for large $|x_t|$. So the influence of the *speculative bias channel* becomes stronger the further x_t is away from its unconditional mean. This potentially shifts wealth to the first agent, who has the correct beliefs about ρ_x. We observe this pattern in the left panel ($x_t = -0.008$), where for large $\underline{\lambda}_t^2$ the average change in the weights $\underline{\lambda}_{t+1}^2 - \underline{\lambda}_t^2$ becomes negative. For positive x_t agent 1 becomes less afraid of long-run risks and hence is less willing to pay to insure against them. Therefore, the average increase in the weights of agent 2 decrease for $x_t = 0.0013$ compared to $x_t = 0$. For very large x_t (right panel) the influence of the *speculative bias channel* dominates and hence the results reverse. The second agent wins if there is a negative shock (blue line), but loses if there is a positive shock (yellow line). The *risk premium channel* becomes negligible and the second agent loses on average as she bets on states that have a vanishing probability under the true measure (see Figure 3). So, the *risk premium channel* dominates the *speculative bias channel* for x_t close to its unconditional mean; only for very large x_t the *speculative bias channel* dominates and then agent 2 potentially loses wealth (on average). However, values of $x_t = 0.008$ (+4 standard deviation of x_t) occur only very rarely; most of the time, the process stays within the range where the *risk premium channel* clearly dominates the *speculative bias channel* and so, on average, agent 2's consumption share increases.

In Figure 5 we show the corresponding results for $\rho_x^2 = 0.95$ instead of $\rho_x^2 = 0.975$. The decrease in ρ_x^2 increases the influence of the *speculative bias channel* as the beliefs of the second agent are "more wrong" on average and hence will shift wealth to the first investor. Furthermore, the second agent is less afraid of long-run risks and therefore will be willing to sell more insurance. So, the influence of the *risk premium channel* also increases, which—on the other hand—shifts wealth to the second investor. Looking at the aggregate effects, we observe that for $x_t = 0$ the change in the weights $\underline{\lambda}_{t+1}^2 - \underline{\lambda}_t^2$ becomes larger on average. (Note the different scale. For a better visualization we show the average change separately in Figure 12 in Appendix C.) This increase reflects the increasing influence of the *risk premium channel* compared to the case with $\rho_x^2 = 0.975$. However, for larger $|x_t|$, the influence of the *speculative bias channel* quickly increases and only for small $\underline{\lambda}_t^2$—where there is a large share of investors who want to buy insurance against long-run risks—the *risk premium channel* dominates. This observation explains why the median consumption share in Figure 2 only increases to a certain level and does not converge further towards 1. The magnitude of the change in the weights explains the large drops and recoveries that we observe in Figure 2. For example, for the extreme case with $x_t = -0.008$ a large negative shock implies a drop in the weights of more than 0.3 for $\underline{\lambda}_t^2 = 0.5$. This implies a decrease in the consumption share of the second agent of more than 0.3. But as the influence of the *risk premium channel* increases for small $\underline{\lambda}_t^2$ the second agent recovers rather quickly, as can be observed from Figure 2.

4.1.3 The Savings Channel

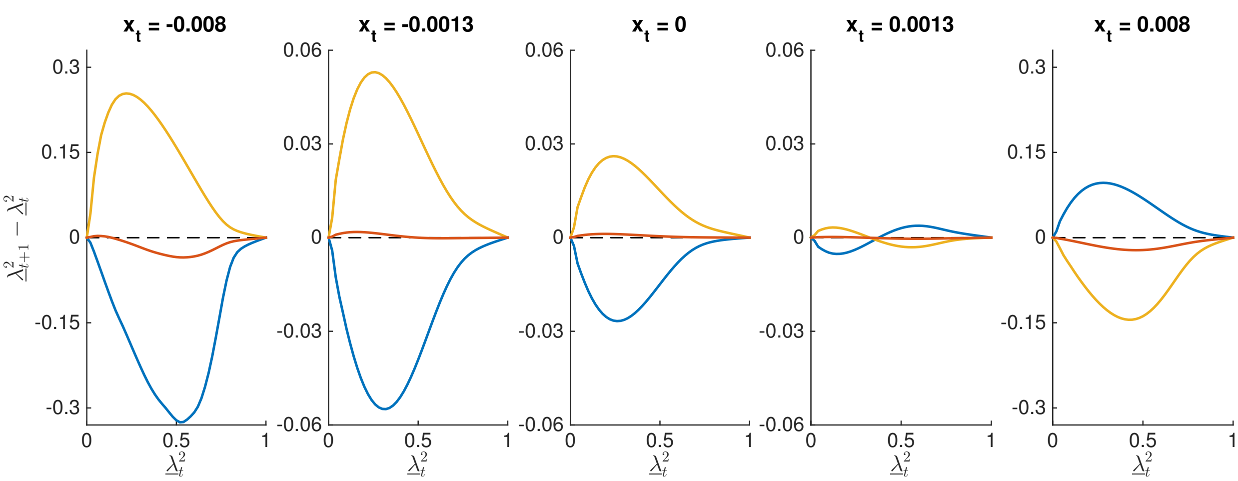
The third channel that influences equilibrium outcomes for Epstein–Zin preferences is the *savings channel*. It states that agents with high subjective beliefs about expected returns will choose a high (low) savings rate if the EIS is large (small). In the long-run risk model the EIS needs to be significantly larger than 1 in order to model a strong preference for the early resolution of risks. Hence, the agent with the higher subjective expected returns chooses

Figure 4: Changes in the Wealth Distribution—The Epstein–Zin Case



The figure shows the change in the optimal weights $\underline{\lambda}^2_{t+1} - \underline{\lambda}^2_t$ as a function of $\underline{\lambda}^2_t$. From left to right, the change is shown for $x_t = \{-0.008, -0.0013, 0, 0.0013, 0.008\}$ (± 4 standard deviations). The blue line depicts the case of a negative shock in x_{t+1} ($x_{t+1} - \rho_x x_t = -0.001$) and the yellow line of a positive shock in x_{t+1} ($x_{t+1} - \rho_x x_t = 0.001$). The red line shows the average over all shocks. Baseline calibration with $\rho_x = \rho^1_x$.

Figure 5: Changes in the Wealth Distribution—The Epstein–Zin Case ($\rho^2_x = 0.95$)



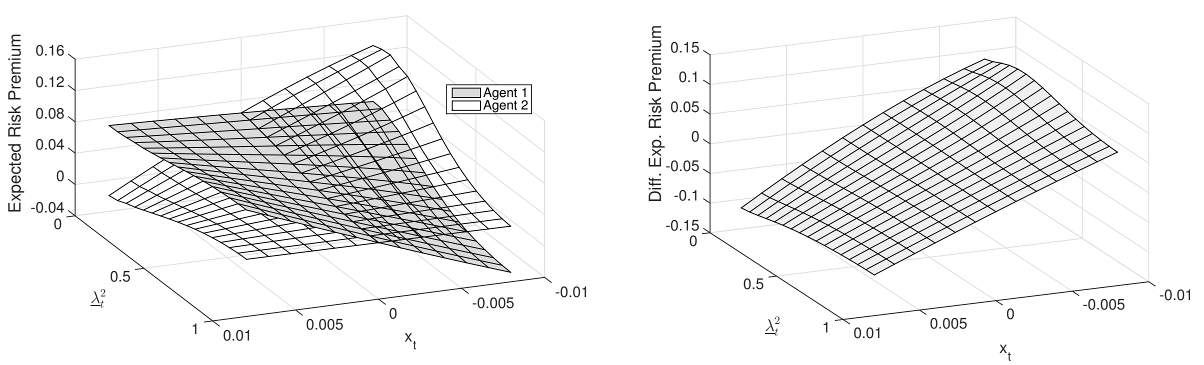
The figure shows the change in the optimal weights $\underline{\lambda}^2_{t+1} - \underline{\lambda}^2_t$ as a function of $\underline{\lambda}^2_t$. From left to right, the change is shown for $x_t = \{-0.008, -0.0013, 0, 0.0013, 0.008\}$ (± 4 standard deviations). The blue line depicts the case of a negative shock in x_{t+1} ($x_{t+1} - \rho_x x_t = -0.001$) and the yellow line of a positive shock in x_{t+1} ($x_{t+1} - \rho_x x_t = 0.001$). The red line shows the average over all shocks. Calibration with $\rho_x = \rho^1_x = 0.985$ and $\rho^2_x = 0.95$.

a higher savings rate and therefore—everything else being equal—this agent's consumption share increases relative to the agent with the lower expected returns.

Figure 6 shows the subjective expected risk premia of the two agents as a function of the states (Figure 6a) as well as the difference between the two risk premia (Figure 6b). Agent 2 has higher subjective risk premia for small x_t and the opposite is true for large x_t. Therefore, for small (large) x_t, agent 2 will choose a higher (lower) savings rate compared to agent 1. However, we find that in the aggregate, the influence of the *savings channel* is rather small compared to that of the *risk premium channel* and the *speculative bias channel*. In Figure 7 we show the corresponding results to Figure 4, but with $\psi^1 = \psi^2 = 1.1$ instead of $\psi^1 = \psi^2 = 1.5$ and, hence, a smaller influence of the *savings channel*. We observe that the quantitative change is rather small and that the qualitative conclusions stay the same.

Figure 6: Expected Subjective Risk Premia

(a) (b)

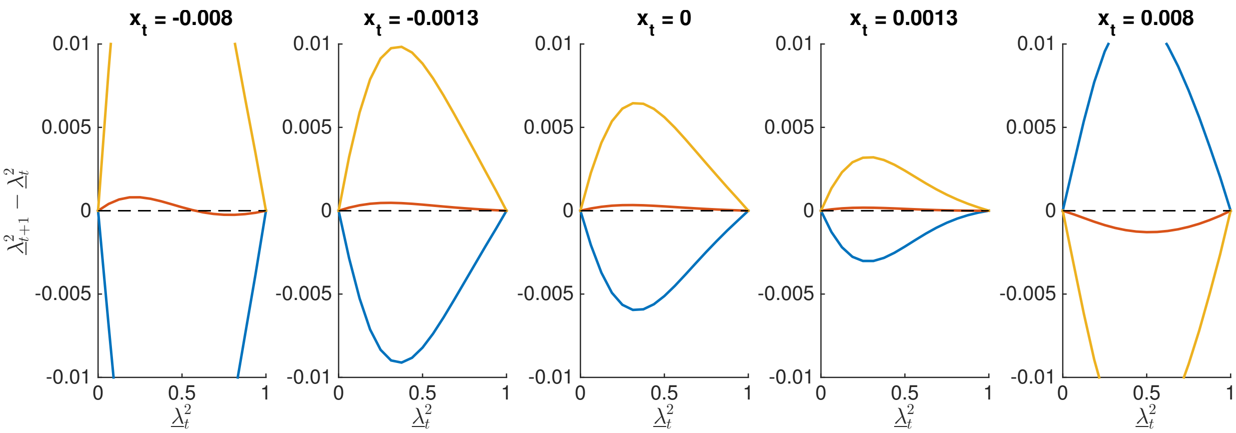


The figure shows the expected subjective risk premium of the two agents as a function of the states $\underline{\lambda}_t^2$ and x_t. Panel (a) shows the premia for the two agents and Panel (b) the difference between the subjective risk premia of agent 2 and agent 1. Baseline calibration with $\rho_x = \rho_x^1 = 0.985$ and $\rho_x^2 = 0.975$.

4.2 Examination of the Risk Premium Channel (Robustness of the Results)

In this section we examine the influence of the *risk premium channel* in more detail. We have argued that, if risk premia are high, the influence of the *risk premium channel* is strong. This will in turn shift wealth to those investors who are skeptical about the presence of long-run risks. In Figure 8a we show the median consumption share of agent 2 (as in Figure 1) for different degrees of risk aversion $\gamma^h = \{2, 5, 10\}$. For $\gamma^1 = \gamma^2 = 10$ the equity premium for the representative-agent economies either populated only by agent 1 or agent 2 are 6.53%

Figure 7: Changes in the Wealth Distribution—Sensitivity of the Epstein–Zin Case ψ^h



The figure shows the change in the optimal weights $\lambda^2_{t+1} - \lambda^2_t$ as a function of λ^2_t. From left to right, the change is shown for $x_t = \{-0.008, -0.0013, 0, 0.0013, 0.008\}$ ($\pm$ 4 standard deviations). The blue line depicts the case of a negative shock in x_{t+1} ($x_{t+1} - \rho_x x_t = -0.001$) and the yellow line of a positive shock in x_{t+1} ($x_{t+1} - \rho_x x_t = 0.001$). The red line shows the average over all shocks. Baseline calibration with $\rho_x = \rho^1_x$ and $\psi^1 = \psi^2 = 1.1$ (instead of 1.5 as in the baseline model).

and 2.76%, respectively (see Table 3). For a risk aversion of $\gamma^1 = \gamma^2 = 5$ they decrease to 2.71% and 0.72% and for $\gamma^1 = \gamma^2 = 2$ the premia are only -0.61% and -0.68%. So for $\gamma^h = 5$ and $\gamma^h = 2$ we expect the impact of the *risk premium channel* to decrease significantly. For $\gamma^h = 10$ (yellow line) the influence of the *risk premium channel* is strong. Hence, agent 2 profits from selling the insurance against long-run risks and rapidly accumulates wealth. For $\gamma^h = 5$ (red line) this effect becomes less severe and her consumption share increases less quickly. For $\gamma^h = 2$ (blue line) risk premia are negative; the *risk premium channel* has no influence and the *speculative bias channel* dominates equilibrium outcomes. As $\rho_x = \rho^1_x$, the *speculative bias channel* works in favor of agent 1 (agent 2 bets on states that have a vanishing probability under the true probability measure) and agent 1 dominates the economy in the long run. If agent 2 has the correct beliefs $\rho_x = \rho^2_x$, the *speculative bias channel* works in favor of agent 2. We show this case in Figure 8b. The blue line shows the consumption shares for $\rho_x = \rho^1_x$ and the red line for $\rho_x = \rho^2_x$. So in the absence of the *risk premium channel*, the *speculative bias channel* determines equilibrium outcomes.

In Figure 8c we depict the robustness of our findings with regard to the level of the persistences of x_t. We show the consumption paths for $\rho^2_x = 0.6$ and $\rho^1_x = 0.5$ instead of for 0.975 and 0.985, respectively. Lowering the persistence will—similarly to the decrease in risk aversion—bring down the equity premium to -0.74%. Consequently, we observe that in this setup the dynamics of the consumption shares strongly depend on the true value of ρ_x as the *speculative bias channel* dominates—that is, the agent with the correct beliefs will dominate the economy.

But long-run risk models require a high degree of risk aversion and a high persistence level of the long-run risk process in order to obtain an equity premium consistent with the data. Consequently, the impact of the *risk premium channel* will be strong and those investors who are skeptical about the presence of long-run risks will dominate the economy. The qualitative implications also hold irrespective of the true value of the underlying persistence of the long-run risk process. In Figure 9 we show the consumption paths with $\rho_x^1 = 0.985$ and $\rho_x^2 = 0.975$ for different values of $\rho_x = \{0, 0.9, 0.99\}$. A lower persistence of ρ_x implies that x_t will remain closer to its unconditional mean (given the same standard deviation). As Figure 4 shows, for x_t close to 0, the consumption share of the second agent increases on average. Hence, the lower the true persistence, the faster the increase in the consumption share. But even for the very large value of ρ_x of 0.99, the risk premium effect still dominates and the second agent dominates the economy in the long run.

4.3 Correcting for the Difference in Mean Consumption Growth

Different beliefs about the persistence of the long-run risk process imply that—everything else being equal—the agent also has different beliefs about the mean of the gross growth rate of consumption $E\left(\frac{C_{t+1}}{C_t}\right)$ due to Jensen's inequality. In this section we show that our results are not driven by this simple mean effect, but rather by the time varying *risk premium channel* as demonstrated in the previous section. In fact, when we correct for the belief difference in the mean growth rate of consumption, the consumption share of the skeptical investor increases even faster. For the long-run risks model (15), the mean growth rate of consumption is given by

$$E\left(\frac{C_{t+1}}{C_t}\right) = E\left(e^{\Delta c_{t+1}}\right) = e^{\mu_c + 0.5\sigma^2 + 0.5\frac{\phi_x^2\sigma^2}{1-(\rho_x)^2}}. \tag{18}$$

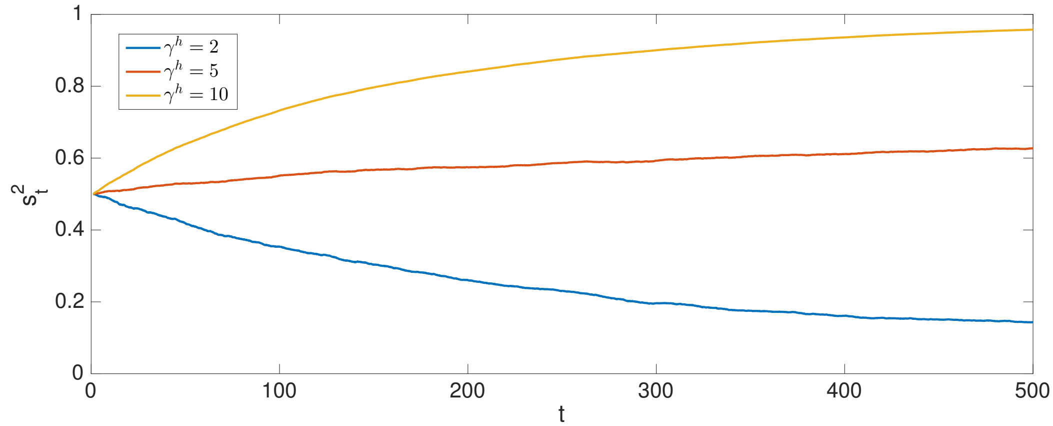
For $\rho_x^2 < \rho_x^1 = \rho_x$ we have that

$$E^2\left(\frac{C_{t+1}}{C_t}\right) = e^{\mu_c + 0.5\sigma^2 + 0.5\frac{\phi_x^2\sigma^2}{1-(\rho_x^2)^2}} < E\left(\frac{C_{t+1}}{C_t}\right). \tag{19}$$

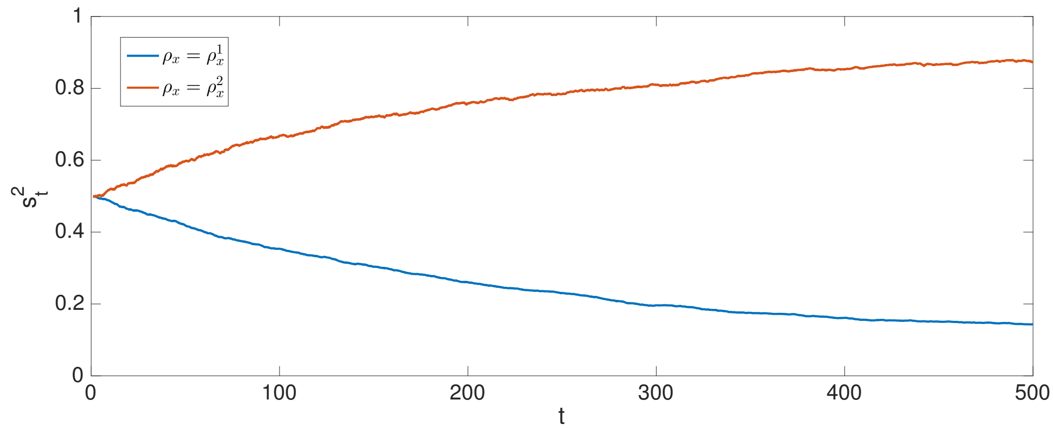
So the second agent believes in a lower mean growth rate of consumption as she believes in a lower persistence and hence a lower unconditional volatility of the long-run risk process. We correct for this belief difference by setting the subjective belief of the second agent with regard to mean log consumption growth to $\mu_c^2 = \mu_c + 0.5\frac{\phi_x^2\sigma^2}{1-(\rho_x)^2} - 0.5\frac{\phi_x^2\sigma^2}{1-(\rho_x^2)^2}$. Once we correct for this difference, the consumption shares of the skeptical investor increase even faster. For

Figure 8: The Risk Premium and Speculative Bias Channels

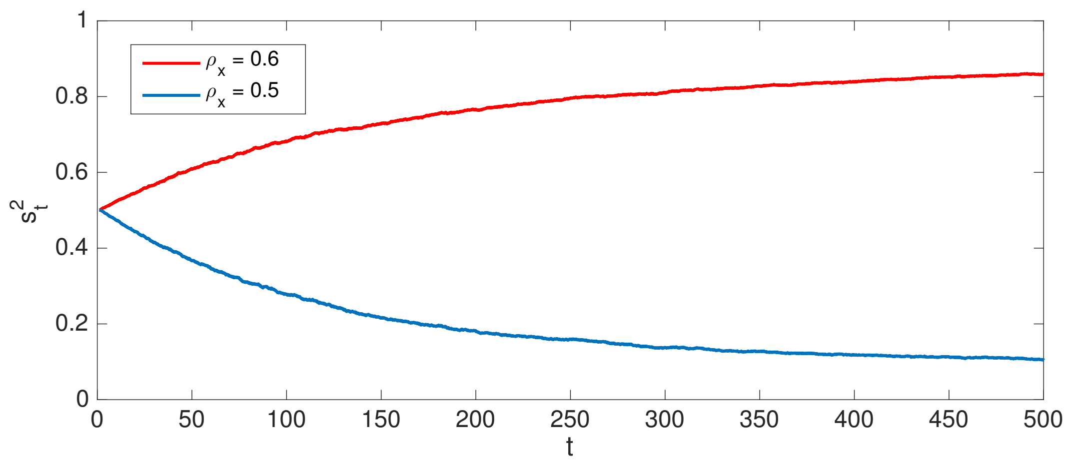
(a) $\rho_x^1 = 0.985$, $\rho_x^2 = \rho_x = 0.975$



(b) $\gamma^h = 2$, $\rho_x^1 = 0.985$, $\rho_x^2 = 0.975$

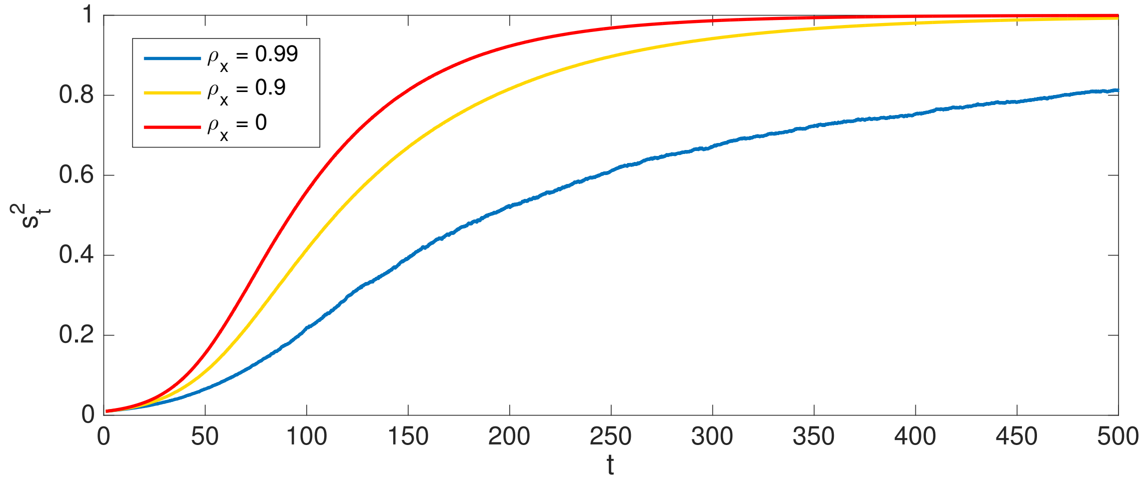


(c) $\gamma^h = 10$, $\rho_x^2 = 0.6$, $\rho_x^1 = 0.5$



The figure shows the median consumption share of agent 2 for 1'000 samples each consisting of 500 years of simulated data. Panel (a) shows the time series for different degrees of risk aversion $\gamma^h \in \{2, 5, 10\}$. Agent 2 believes that $\rho_x^2 = 0.975$ and agent 1 has the correct beliefs with $\rho_x^1 = \rho_x = 0.985$. Panel (b) shows the time series for $\gamma^h = 2$, $\rho_x^1 = 0.985$, and $\rho_x^2 = 0.975$ for the two cases in which either agent 1 (blue line) or agent 2 (red line) has the correct beliefs. Panel (c) shows the time series for $\gamma^h = 10$, $\rho_x^1 = 0.6$, and $\rho_x^2 = 0.5$ for the two cases where either agent 1 (blue line) or agent 2 (red line) has the correct beliefs.

Figure 9: The Risk Premium and Speculative Bias Channels (2)



The figure shows the median consumption share of agent 2 for 1'000 samples each consisting of 500 years of simulated data. Both agents have a risk aversion of $\gamma = 10$. The results are shown for $\rho_x^2 = 0.975$ and $\rho_x^1 = 0.985$ for different values of $\rho_x = \{0, 0.9, 0.99\}$.

the original specification with an initial allocation of $s_0^2 = 0.01$, the consumption shares of the skeptical investor increased to 0.2824, 0.6376, and 0.9278 after 100, 200, and 500 years, respectively (see Table 2). With the corrected mean we obtain values of 0.2827, 0.6379, and 0.9281. Hence, our results are not driven by the effect of different mean beliefs about consumption growth. This result is also in accordance with Borovička (2015), who shows that underestimation of the mean growth rate lowers the chances of survival while overestimation has the opposite effect due to the positive *risk premium channel*. Consequently, in our model specification, the effect of the mean growth rate should lead the skeptical investor to have lower consumption shares. And indeed, once we correct the mean growth rate estimate of the skeptical investor we obtain a faster increase in the consumption shares of that skeptical investor.

5 Heterogeneity in the Preference Parameters

In this section we relax the assumption of identical preferences and analyze the influence of differences in the preference parameters on equilibrium outcomes. Yan (2008) shows that for CRRA preferences and i.i.d. consumption growth differences in beliefs can be offset by only small differences in the preference parameters. We show that in our model setup with Epstein–Zin preferences, very large differences in the EIS are required to obtain variations in the consumption shares similar to the setup with different beliefs. Differences in the risk aversion parameter induce larger changes to the consumption shares, but values for the risk aversions significantly larger than 10—the maximum values considered as reasonable in the literature—are required to offset the effect obtained from the differences in beliefs.

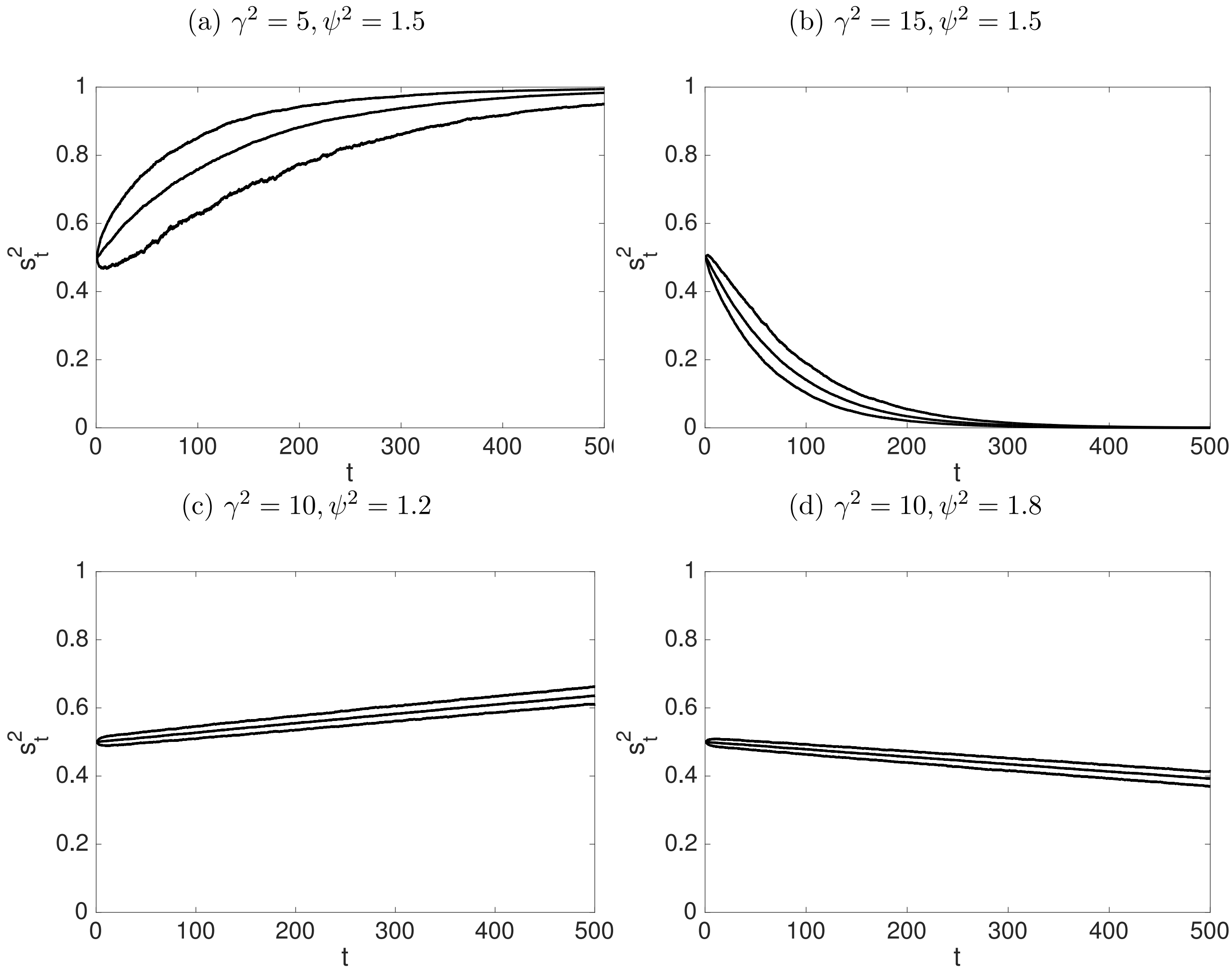
For the analysis, we assume that beliefs are the same among agents but that the agents differ with regard to their degree of risk aversion and their intertemporal elasticity of substitution; and second, we consider model parametrizations with heterogeneity in both beliefs and preferences.

5.1 Identical Beliefs and Heterogeneous Preferences

In the first example, we assume that both investors have identical beliefs about the persistence of x_t with $\rho_x = \rho_x^1 = \rho_x^2 = 0.985$, but they differ with regard to their preference parameters. For agent 1 we use the standard preference parameters from Bansal and Yaron (2004) with a risk aversion of $\gamma^1 = 10$ and an EIS of $\psi^1 = 1.5$. For the second agent we consider different preference parameter combinations. In Figure 10 we show the consumption shares of the second agent for 500 years of simulated data. Panel (a) shows the case in which agent 2 has a smaller degree of risk aversion with $\gamma^2 = 5$, and Panel (b) the case of a larger risk aversion with $\gamma^2 = 15$. We observe that for the case of a lower risk aversion, the consumption shares of the second agent increase rapidly. For the case of a higher risk aversion, we observe the opposite, namely a rapid decrease in the consumption share. The changes can again be explained by the *risk premium channel*. An investor with a higher degree of risk aversion is more afraid of negative shocks to the economy and would therefore like to buy insurance against these risks. As risk premia in the economy are high, the premium this investor is willing to pay is high. The investor with the lower degree of risk aversion, on the other hand, will be willing to provide this insurance as he is less afraid of the long-run risk. Hence, in Panel (a), where agent 2 has a lower risk aversion, she is willing to provide the insurance; as risk premia in the economy are high (the equity premia in the economies populated by agent 1 or agent 2 alone are 6.53% and 2.71%, respectively), she benefits on average. In Panel (b) agent 2 is a lot more risk averse than agent 1 and hence the *risk premium channel* works against agent 2. As risk premia in Panel (b) are larger (the equity premium in the economies populated by agent 1 or agent 2 alone are 6.53% and 9.49%, respectively), we observe a faster decrease in Panel (b) than the increase in Panel (a), with a median share after 100 years of only 14% for Panel (b) and 75% for Panel (a).

In Panels (c) and (d) we show the case of the two agents having identical levels of risk aversion but different EIS parameters. In Panel (c) agent 2 has a smaller EIS compared to

Figure 10: Consumption Shares for Heterogeneous Preferences: Simulations



The figure shows the median, 5%, and 95% quantile paths of the consumption share of agent 2 for 1'000 samples each consisting of 500 years of simulated data. The investors have identical beliefs about ρ_x with $\rho_x = \rho_x^1 = \rho_x^2 = 0.985$, but differ with respect to their preference parameters. Agent 1 has a risk aversion of $\gamma^1 = 10$ and an EIS of $\psi^1 = 1.5$ and the different panels show the results for different preference parameters of the second investor.

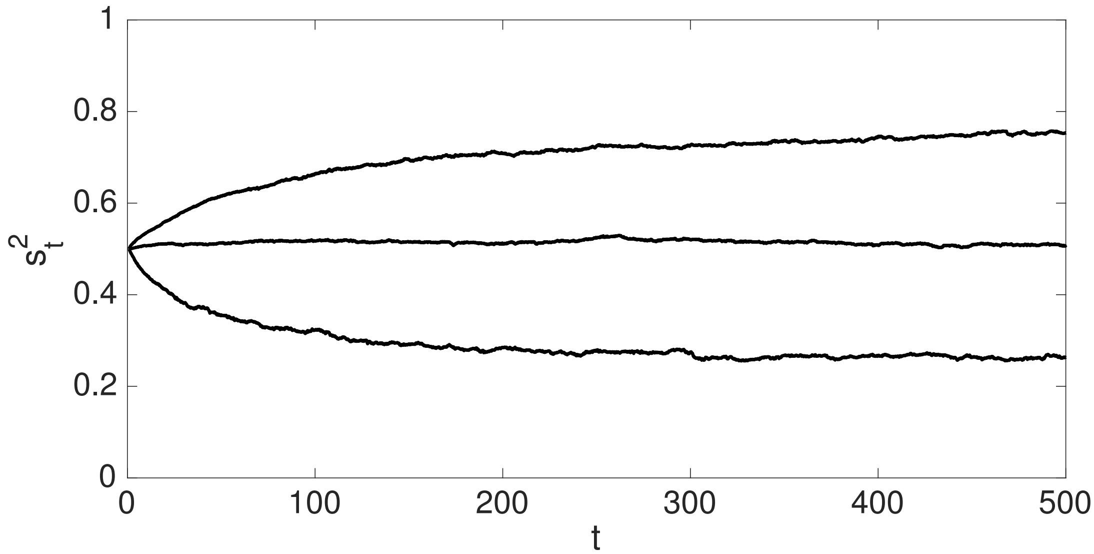
agent 1, $\psi^2 = 1.2$, and in Panel (d) she has a larger EIS, $\psi^2 = 1.8$. We observe that the consumption share of the agent with the lower EIS increases on average; but the change in the shares is much slower compared to the case with different degrees of risk aversion and there is also no sharp increase or decrease initially. As the *risk premium channel* is the most important channel for the change in the consumption shares and as the difference in the EIS has a negligible influence on risk premia (the equity premium of the economy only populated by agent 2 is 6.04% for $\psi^2 = 1.2$ and 6.95% for $\psi^2 = 1.8$), the influence of the difference in the EIS on the change in the consumption shares is rather small. As a robustness check, we show in Figure 13 in Appendix C the corresponding results for two more extreme cases: $\psi^2 = 0.8$ and $\psi^2 = 2.2$. The consumption shares still move very slowly. Therefore, we conclude that preference heterogeneity in the EIS has only a minor influence in the long-run risk model—in particular when compared to the effects of heterogeneity in beliefs and in the risk aversion parameter.

5.2 Heterogeneous Beliefs and Preferences

In Figure 11 we show the evolution of the consumption shares in an economy in which the investors have both different beliefs about the persistence of x_t and different preference parameters. The first agent has the same characteristics as in the baseline economy; he is the long-run risk investor of Bansal and Yaron (2004), who has the correct beliefs with $\rho_x = \rho_x^1 = 0.985$, a risk aversion of $\gamma^1 = 10$, and an EIS of $\psi^1 = 1.5$. The second agent believes that $\rho_x^2 = 0.975$ and has an EIS of $\psi^2 = 1.5$; we choose γ^2 so that there is no visible trend in the average consumption shares of the two agents. We find that for a value of $\gamma^2 \approx 17.5$ the effect of a larger risk aversion and the belief in a lower value of ρ_x cancel each other so that the median consumption share in Figure 11 remains approximately at the same level.[6] For the representative-agent economy with a risk aversion $\gamma^2 = 17.5$ and $\rho_x^2 = 0.975$, the equity premium is 5.6% and therefore only slightly smaller than the 6.53% for the representative-agent economy of the first agent. So, in this setup the equity premium would not collapse in the two-agent economy. However, even for the small belief difference in ρ_x, a large degree of risk aversion is needed so that the two effects cancel each other. As it is usually assumed that the risk aversion coefficient is not significantly larger than 10, simply increasing the risk aversion of the investor who believes in a smaller persistence of x_t does not seem like a plausible

[6]We tried to obtain similar results where we increase the EIS instead of the risk aversion accordingly. However, as Table 10 shows, the effect of ψ^2 on the equilibrium consumption shares is so small that a very large value of ψ^2 would be needed, which makes solving the model computationally very challenging.

Figure 11: Consumption Shares for Heterogeneous Preferences and Beliefs—Simulations



The figure shows the median, 5%, and 95% quantile paths of the consumption share of agent 2 for 1'000 samples each consisting of 500 years of simulated data. Agent 2 has a risk aversion of $\gamma^2 = 17.5$ and believes that $\rho_x^2 = 0.975$ and agent 1 has the correct beliefs about ρ_x with $\rho_x = \rho_x^1 = 0.985$ but a lower risk aversion of $\gamma^1 = 10$. Both agents have the same EIS of $\psi^1 = \psi^2 = 1.5$.

mechanism via which to retain the large equity premium in the two-agent economy.

6 Conclusion

We have performed a detailed study of heterogeneity in agents' beliefs for the long-run risk model of Bansal and Yaron (2004). In particular, we consider agents with different beliefs about the level of persistence of long-run risk. We find that as long as the level of heterogeneity is not too large, agents who believe in a lower level of persistence come to dominate the economy rather quickly relative to agents who believe in a higher level of persistence. This holds even if the agent with the higher level of persistence holds the correct belief. This suggests that for long-run risk to work as an explanation of the equity premium, it is insufficient for long-run risk in consumption to merely *exist*—agents must also *all agree* on the amount of long-run risk the economy experiences.

For larger differences in beliefs, both agents survive in the long run. In that case, belief heterogeneity leads to considerable volatility in asset prices. This result suggests that—even though belief heterogeneity does not explain the equity premium—it can explain the large asset price volatility found in the data.

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Appendix

A Proofs and Details

In this appendix, we provide proofs for the theoretical results presented in Section 2. Along the way, we derive a system of first-order conditions for Epstein–Zin preferences. This system constitutes the foundation for our numerical solution method (see Appendix B).

A.1 Proofs for Section 2.1

Proof of Theorem 1. Let $\boldsymbol{\lambda} = \{\bar{\lambda}^1, \ldots, \bar{\lambda}^H\}$ be a set of Negishi weights and let $\{\boldsymbol{C}\}_0 = \{\{C^1\}_0, \ldots, \{C^H\}_0\}$ denote a vector of agents' consumption processes. The optimal decision $\{\boldsymbol{C}\}_0^*$ of the social planner in the initial period assigns consumption streams to all individual agents for all periods and possible states. Obviously, the optimal decisions must satisfy the market-clearing condition (1) in all periods and states. For ease of notation we again abbreviate the state dependence; we use C_t^h for $C^h(y^t)$ and $U_{\{t\}}^h$ for $U^h\left(\{C^h\}_t\right)$.

To derive the first-order conditions, we borrow a technique from the calculus of variations. For any function f_t, we can vary the consumption of two agents by

$$\begin{aligned} C_t^h &\rightarrow \quad C_t^h + \epsilon f_t \\ C_t^l &\rightarrow \quad C_t^l - \epsilon f_t. \end{aligned} \tag{20}$$

It is sufficient to consider the variation with $l = 1$ and $h \in \mathbb{H}^-$. For an optimal allocation it must be true that

$$\left.\frac{\mathrm{d}SP(\{\boldsymbol{C}\}_0; \boldsymbol{\lambda})}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0. \tag{21}$$

This gives us

$$\bar{\lambda}^h \hat{U}_{0,t}^h = \bar{\lambda}^1 \hat{U}_{0,t}^1, \qquad h \in \mathbb{H}^-, \tag{22}$$

where $\hat{U}_{t,t+k}^h$ is defined as

$$\hat{U}_{t,t+k}^h = \left.\frac{\mathrm{d}U^h(C_t^h, \ldots, C_{t+k}^h + \epsilon f_{t+k}, \ldots)}{\mathrm{d}\epsilon}\right|_{\epsilon=0}. \tag{23}$$

Using the expression given in Equation (2), the derivative $\hat{U}_{t,t+k}^h$ satisfies a recursive equation with the initial condition

$$\hat{U}^h_{t,t} = \left.\frac{\mathrm{d}U^h(C^h_t + \epsilon f_t, \ldots)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = F^h_1\left(C^h_t, R_t[U^h_{\{t+1\}}]\right) \cdot f_t, \tag{24}$$

where $F^h_k\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right)$ denotes the derivative of $F^h\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right)$ with respect to its kth argument. The recursive step is given by

$$\begin{aligned}
\hat{U}^h_{t,t+k} &= \left.\frac{\mathrm{d}F^h\left(C^h_t, R^h_t\left[U^h(C^h_{t+1}, \ldots C^h_{t+k} + \epsilon f_{t+k}, \ldots)\right]\right)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\
&= F^h_2\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right) \cdot \left.\frac{\mathrm{d}R^h_t\left[U^h(\cdot)\right]}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\
&= F^h_2\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right) \cdot \frac{\mathrm{d}G^{-1}_h\left(E^h_t G_h\left[U^h(\cdot)\right]\right)}{\mathrm{d}E^h_t G_h[U^h(\cdot)]} \cdot \left.\frac{\mathrm{d}E^h_t G_h[U^h(\cdot)]}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\
&= F^h_2\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right) \cdot \frac{1}{G'_h(G^{-1}_h(E^h_t G_h[U^h_{\{t+1\}}]))} \cdot E^h_t\left(G'_h(U^h_{\{t+1\}}) \cdot \hat{U}^h_{t+1,t+k}\right) \\
&= F^h_2\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right) \cdot \frac{E^h_t\left(G'_h(U^h_{\{t+1\}}) \cdot \hat{U}^h_{t+1,t+k}\right)}{G'_h(R^h_t[U^h_{\{t+1\}}])},
\end{aligned} \tag{25}$$

where we use $\frac{\partial G^{-1}(x)}{\partial x} = \frac{1}{G'(G^{-1}(x))}$ and abbreviate $U^h(C^h_{t+1}, \ldots C^h_{t+k} + \epsilon f_{t+k}, \ldots)$ by $U^h(\cdot)$. We can recast this recursion into a useful form. For this purpose, we define a second recursion $U^h_{t,t+k}$ by

$$U^h_{t,t} = F^h_1\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right) \tag{26}$$

and

$$U^h_{t,t+k} = \Pi^h_{t+1} \cdot U^h_{t+1,t+k}, \tag{27}$$

where

$$\Pi^h_{t+1} = F^h_2\left(C^h_t, R^h_t[U^h_{\{t+1\}}]\right) \cdot \frac{G'_h(U^h_{\{t+1\}})}{G'_h(R^h_t[U^h_{\{t+1\}}])} \frac{\mathrm{dP}^h_{t,t+1}}{\mathrm{dP}_{t,t+1}}. \tag{28}$$

A simple induction shows that

$$\hat{U}^h_{t,t+k} = E_t(U^h_{t,t+k} f_t). \tag{29}$$

Plugging (29) into the optimality condition (22) we obtain

$$E_0\left((\bar{\lambda}^h U^h_{0,t} - \bar{\lambda}^1 U^1_{0,t}) f_t\right) = 0, \qquad h \in \mathbb{H}^-. \tag{30}$$

Under a broad range of regularity conditions, this condition implies that

$$\bar{\lambda}^h U_{0,t}^h = \bar{\lambda}^1 U_{0,t}^1, \qquad h \in \mathbb{H}^-. \tag{31}$$

For example, if $\bar{\lambda}^h U_{0,t}^h - \bar{\lambda}^1 U_{0,t}^1$ has finite variance, then this holds for the Riesz Representation Theorem for L^2 random variables. We can then split Expression (31) into two parts. First define $\lambda_0^h \equiv \bar{\lambda}^h$ to obtain

$$\frac{\lambda_0^h}{\lambda_0^1} = \frac{U_{0,t}^1}{U_{0,t}^h} = \frac{\Pi_0^1}{\Pi_0^h}\frac{U_{1,t}^1}{U_{1,t}^h} = \frac{\Pi_0^1}{\Pi_0^h}\frac{\lambda_1^h}{\lambda_1^1}, \qquad h \in \mathbb{H}^-,$$

where λ_1^h denotes the Negishi weight in the social planner's optimal solution in $t = 1$. Generalizing this equation for any period t, we obtain the following dynamics for the optimal weight[7] λ_{t+1}^h

$$\frac{\lambda_{t+1}^h}{\lambda_{t+1}^1} = \frac{\Pi_{t+1}^h}{\Pi_{t+1}^1}\frac{\lambda_t^h}{\lambda_t^1}, \qquad h \in \mathbb{H}^-. \tag{32}$$

Inserting the initial condition (26) into (31) for $t = 0$ and generalizing it for any social planner's optimal solution at time t yields

$$\lambda_t^h F_1^h\left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right) = \lambda_t^1 F_1^1\left(C_t^1, R_t^1[U_{\{t+1\}}^1]\right), \qquad h \in \mathbb{H}^-. \tag{33}$$

Equation (33) states the optimality conditions for the individual consumption choices at any time t. This completes the proof of Theorem 1. □

Note that for time-separable utility, $F_1^h\left(C_t^h, R_t^h[U_{\{t+1\}}^h]\right)$ is simply the marginal utility of agent h at time t, and so we obtain the same optimality condition as, for example, Judd, Kubler, and Schmedders (2003) (see Equation (7) on page 2209). In this special case the Negishi weights can be pinned down in the initial period and thereafter remain constant. For general recursive preferences this is not true. The optimal weights vary over time following the law of motion described by Equation (32).

We can use Equations (32) and (33) together with the market-clearing condition (1) to compute the social planner's optimal solution. We therefore define $\boldsymbol{\lambda}_t^- = \{\lambda_t^2, \lambda_t^3, \ldots, \lambda_t^H\}$ and let V^h denote the value function of agent $h \in \mathbb{H}$. We are looking for model solutions of the form $V^h(\boldsymbol{\lambda}_t^-, y^t)$. So, the model solution depends on both the exogenous state y^t and the

[7] Note that we can either solve the model in terms of the ratio $\frac{\lambda_t^h}{\lambda_t^1}$ (this is equal to setting $\lambda_t^1 = 1$ for all t as done in Judd, Kubler, and Schmedders (2003)) or we can normalize the weights so that they remain bounded in $(0, 1)$. Our solution method uses the latter approach as it obtains better numerical properties.

time-varying Negishi weights $\boldsymbol{\lambda}_t^-$. An optimal allocation is then characterized by the following four equations:

- the market-clearing condition (1)

$$\sum_{h=1}^{H} C^h(\boldsymbol{\lambda_t^-}, y^t) = C(y^t); \tag{34}$$

- the value functions (2) of the individual agents

$$V^h(\boldsymbol{\lambda_t^-}, y^t) = F^h\left(C^h(\boldsymbol{\lambda_t^-}, y^t), R_t^h[V^h(\boldsymbol{\lambda_{t+1}^-}, y^{t+1})]\right), \qquad h \in \mathbb{H}; \tag{35}$$

- the optimality conditions (33) for the individual consumption decisions for $h \in \mathbb{H}^-$

$$\lambda_t^h F_1^h\left(C^h(\boldsymbol{\lambda_t^-}, y^t), R_t^h[V^h(\boldsymbol{\lambda_{t+1}^-}, y^{t+1})]\right) = \lambda_t^1 F_1^1\left(C^1(\boldsymbol{\lambda_t^-}, y^t), R_t^1[V^1(\boldsymbol{\lambda_{t+1}^-}, y^{t+1})]\right); \tag{36}$$

- the equations (32) for the dynamics of $\boldsymbol{\lambda_t^-}$

$$\frac{\lambda_{t+1}^h}{\lambda_{t+1}^1} = \frac{\Pi_{t+1}^h}{\Pi_{t+1}^1}\frac{\lambda_t^h}{\lambda_t^1}, \qquad h \in \mathbb{H}^-, \tag{37}$$

  with

$$\Pi_{t+1}^h = F_2^h\left(C^h(\boldsymbol{\lambda_t^-}, y^t), R_t^h[V^h(\boldsymbol{\lambda_{t+1}^-}, y^{t+1})]\right) \cdot \frac{G_h'(V^h(\boldsymbol{\lambda_{t+1}^-}, y^{t+1}))}{G_h'(R_t^h[V^h(\boldsymbol{\lambda_{t+1}^-}, y^{t+1})])}\frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}. \tag{38}$$

This concludes the general description of the equilibrium obtained from the social planner's optimization problem.

To prove Theorem 2, we first derive a variant of Lemma 1 in Blume and Easley (2006).

Lemma 1. *Let X_t^i, $i = 1, 2, \ldots, H$, be a family of positive random variables for each $t = 0, 1, 2, \ldots$, such that $A \le \sum_i X_t^i \le B$ with $B \in \mathbb{R}_{++}$. Let $f^i : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $i = 1, 2, \ldots, H$, be a family of decreasing functions such that $f^i(x) \to \infty$ as $x \to 0$. If $f^i(X_t^i)/f^j(X_t^j) \to \infty$, then $X_t^i \to 0$ for $t \to \infty$. If $X_t^i \to 0$, then for at least one j, $\limsup_t f^i(X_t^i)/f^j(X_t^j) = \infty$.*

Proof. Since X_t^i is positive, $X_t^i \le B$ for all i, t. By assumption, $0 < f^j(B) \le f^j(X_t^j)$. Thus, $f^i(X_t^i)/f^j(X_t^j) \to \infty$ if and only if $f^i(X_t^i) \to \infty$, which happens when $X_t^i \to 0$ as $t \to \infty$.

Conversely, assume $X_t^i \to 0$. Every period, for at least one j, $X_t^j \ge A/H$ (otherwise $\sum_{i=1}^H X_t^i < A$). Since there are only finitely many random variables, for at least one j we have

$X_t^j \geq A/H$ infinitely often. Then, by assumption, $f^j(X_t^j) \leq f^j(A/H)$ infinitely often, and so $\limsup f^i(X_t^i)/f^j(X_t^j) = \infty$. $\square$

Proof of Theorem 2. By the first-order condition (5), $\lambda_t^j/\lambda_t^i = F_1^i(C_t^i, R_t^i)/F_1^j(C_t^j, R_t^j)$. Since F^h is additively separable, F_1^h is a function of consumption alone. Let $f^i = F_1^i, f^j = F_1^j$, $A = \underline{C}$, and $B = \overline{C}$, and apply Lemma 1. $\square$

A.2 Proofs for Section 2.2

In this section we provide the specific expressions for V^h, F_1^h, F_2^h, and Π^h when the heterogeneous investors have recursive preferences as in Epstein and Zin (1989) and Weil (1989). The value function for Epstein–Zin (EZ) preferences is given by[8]

$$V_t^h = \left[(1-\delta^h)(C_t^h)^{\rho^h} + \delta^h R_t^h \left(V_{t+1}^h\right)^{\rho^h}\right]^{\frac{1}{\rho^h}} \tag{39}$$

with

$$\begin{aligned} R_t^h\left(V_{t+1}^h\right) &= G_h^{-1}\left(E_t^h\left[G_h(V_{t+1}^h)\right]\right) \\ G_h(V_{t+1}^h) &= \left(V_{t+1}^h\right)^{\alpha^h}. \end{aligned}$$

Recall that the parameter δ^h is the discount factor, $\rho^h = 1 - \frac{1}{\psi^h}$ determines the EIS, ψ^h, and $\alpha^h = 1 - \gamma^h$ determines the relative risk aversion γ^h of agent h. The derivatives of $F^h\left(C_t^h, R_t^h[V_{t+1}^h]\right) = V_t^h$ with respect to its first and second argument are then given by

$$F_{1,t}^h = (1-\delta^h)(C_t^h)^{\rho^h-1}(V_t^h)^{1-\rho^h} \tag{40}$$

and

$$F_{2,t}^h = \delta^h R_t^h\left(V_{t+1}^h\right)^{\rho^h-1}(V_t^h)^{1-\rho^h}. \tag{41}$$

In this paper we focus on growth economies. Therefore, we introduce the following normalization to obtain a stationary formulation of the model. We define the consumption share of agent h by $s_t^h = \frac{C_t^h}{C_t}$ and the normalized value functions, $v_t^h = \frac{V_t^h}{C_t}$. Recall that $\Delta c_{t+1} = c_{t+1} - c_t$ with $c_t = \log(C_t)$. The value function (39) is then given by

$$v_t^h = \left[(1-\delta^h)(s_t^h)^{\rho^h} + \delta^h R_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\rho^h}\right]^{\frac{1}{\rho^h}}. \tag{42}$$

[8] For ease of notation, we again abbreviate the dependence on the exogenous state y_t and the endogenous state $\boldsymbol{\lambda}_t^-$. Hence we write V_t^h for $V^h(\boldsymbol{\lambda}_t^-, y_t)$ or C_t^h for $C^h(\boldsymbol{\lambda}_t^-, y_t)$.

By inserting (40) into (36) we obtain the optimality condition for the individual consumption decisions

$$\lambda_t^h F_1^h \left(C^h(\boldsymbol{\lambda}_t^-, y^t), R_t^h[V^h(\boldsymbol{\lambda}_{t+1}^-, y^{t+1})]\right) = \lambda_t^1 F_1^1 \left(C^1(\boldsymbol{\lambda}_t^-, y^t), R_t^1[V^1(\boldsymbol{\lambda}_{t+1}^-, y^{t+1})]\right),$$

which simplifies to

$$\lambda_t^h(1-\delta^h)(C_t^h)^{\rho^h-1}(V_t^h)^{1-\rho^h} = \lambda_t^1(1-\delta^1)(C_t^1)^{\rho^1-1}(V_t^1)^{1-\rho^1}. \tag{43}$$

Recall the definition of the normalized Negish weights, $\underline{\lambda}_t^h = \frac{\lambda_t^h}{(v_t^h)^{\rho^h-1}}$. From Equation (43) we obtain

$$\underline{\lambda}_t^h(1-\delta^h)(s_t^h)^{\rho^h-1} = \underline{\lambda}_t^1(1-\delta^1)(s_t^1)^{\rho^1-1}. \tag{44}$$

This equation is the optimality condition for the individual consumption decisions we employ for solving for the model with Epstein–Zin preferences. Inserting the de-trended weight $\underline{\lambda}_t^h$ into the dynamics for the weights (37), we obtain

$$\frac{\lambda_{t+1}^h}{\lambda_{t+1}^1} = \frac{\underline{\lambda}_{t+1}^h(v_{t+1}^h)^{\rho^h-1}}{\underline{\lambda}_{t+1}^1(v_{t+1}^1)^{\rho^1-1}} = \frac{\underline{\lambda}_t^h(v_t^h)^{\rho^h-1}}{\underline{\lambda}_t^1(v_t^1)^{\rho^1-1}}\frac{\Pi_{t+1}^h}{\Pi_{t+1}^1}, \qquad h \in \mathbb{H}^-. \tag{45}$$

Plugging the expressions for Epstein–Zin preferences (39)–(41) into Equation (38), we obtain the following expression for Π_{t+1}^h:

$$\begin{aligned}\Pi_{t+1}^h &= \delta^h R_t^h \left(V_{t+1}^h\right)^{\rho^h-1}(V_t^h)^{1-\rho^h}\frac{\left(V_{t+1}^h\right)^{\alpha^h-1}}{R_t^h\left(V_{t+1}^h\right)^{\alpha^h-1}}\frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}} \\ &= \delta^h(V_t^h)^{1-\rho^h}\frac{\left(V_{t+1}^h\right)^{\alpha^h-1}}{R_t^h\left(V_{t+1}^h\right)^{\alpha^h-\rho^h}}\frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}.\end{aligned} \tag{46}$$

Using the normalized value function $v_t^h = \frac{V_t^h}{C_t}$, we have

$$\Pi_{t+1}^h = \delta^h(v_t^h)^{1-\rho^h}\frac{\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h-1}}{R_t^h\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h-\rho^h}}\frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}. \tag{47}$$

Equation (45) can then be written as

$$\frac{\underline{\lambda}_{t+1}^h}{\underline{\lambda}_{t+1}^1} = \frac{\underline{\lambda}_t^h}{\underline{\lambda}_t^1}\frac{\underline{\Pi}_{t+1}^h}{\underline{\Pi}_{t+1}^1}, \qquad h \in \mathbb{H}^-, \tag{48}$$

where

$$\underline{\Pi}_{t+1}^h = \underbrace{\delta^h e^{\rho^h \Delta c_{t+1}} \frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}}}_{\text{CRRA-Term}} \underbrace{\frac{\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h - \rho^h}}{R_t^h \left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h - \rho^h}}}_{\text{New EZ-Term}}. \tag{49}$$

For $\alpha^h = \rho^h$, we obtain the standard term for CRRA preferences; the dynamics of $\underline{\lambda}_{t+1}^h$ only depend on the subjective discount factor, the EIS, and the subjective beliefs of the investors. For Epstein–Zin preferences, we obtain an extra term that reflects the time trade-off. Using the normalization $\sum_{h=1}^H \underline{\lambda}_t^h = 1$, the dynamics for $\underline{\lambda}_{t+1}^h$ are then given by

$$\underline{\lambda}_{t+1}^h = \frac{\underline{\lambda}_t^h \underline{\Pi}_{t+1}^h}{\sum_{h=1}^H \underline{\lambda}_t^h \underline{\Pi}_{t+1}^h}. \tag{50}$$

Hence, for Epstein–Zin preferences we obtain the following system for the first-order conditions (34)-(38):

The market-clearing condition:

$$\sum_{h=1}^H s_t^h = 1. \tag{MC}$$

The optimality condition for the individual consumption decisions:

$$\underline{\lambda}_t^h (1 - \delta^h)(s_t^h)^{\rho^h - 1} = \underline{\lambda}_t^1 (1 - \delta^1)(s_t^1)^{\rho^1 - 1}, \quad h \in \mathbb{H}^-, \tag{CD}$$

with $\sum_{h=1}^H \underline{\lambda}_t^h = 1$.

The value functions of the individual agents:

$$v_t^h = \left[(1 - \delta^h)(s_t^h)^{\rho^h} + \delta^h R_t^h \left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\rho^h}\right]^{\frac{1}{\rho^h}}, \quad h \in \mathbb{H}. \tag{VF}$$

The equation for the dynamics of $\underline{\lambda}_t^h$:

$$\begin{aligned} \underline{\lambda}_{t+1}^h &= \frac{\underline{\lambda}_t^h \underline{\Pi}_{t+1}^h}{\sum_{h=1}^H \underline{\lambda}_t^h \underline{\Pi}_{t+1}^h} \\ \underline{\Pi}_{t+1}^h &= \delta^h e^{\rho^h \Delta c_{t+1}} \frac{\mathrm{dP}_{t,t+1}^h}{\mathrm{dP}_{t,t+1}} \frac{\left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h - \rho^h}}{R_t^h \left(v_{t+1}^h e^{\Delta c_{t+1}}\right)^{\alpha^h - \rho^h}}, \quad h \in \mathbb{H}^-. \end{aligned} \tag{Dλ}$$

Note that the conditions (MC, CD, VF, Dλ) are just the equilibrium conditions (11)–(14) stated in Section 2.2. We observe that Equation (CD) and hence the individual consumption decisions s_t^h only depend on time t information and that there is no intertemporal dependence. This feature allows us to first solve for s_t^h given the current state of the economy, and in a second step to solve for the dynamics of the Negishi weights. Hence, we can separate solving the optimality conditions (11)–(14) into two steps in order to reduce the computational complexity. In Appendix B we describe this approach in detail.

Using condition (CD) we can prove Theorem 3. Recall that $\rho^h = 1 - \frac{1}{\psi^h} < 1$ for all possible values of an agent's EIS, $\psi^h > 0$.

Proof of Theorem 3. Condition (CD) implies

$$\frac{\underline{\lambda}_t^j}{\underline{\lambda}_t^i} = \frac{(1-\delta^i)(s_t^i)^{\rho^i - 1}}{(1-\delta^j)(s_t^j)^{\rho^j - 1}}.$$

Now let $f^i(s) = s^{\rho^i - 1}$, $f^j(s) = s^{\rho^j - 1}$, and $A = B = 1$, and apply Lemma 1. $\square$

B Solution Method

We describe our solution method for asset-pricing models with heterogeneous agents and recursive preferences.

B.1 Computational Procedure—A Two-Step Approach

For ease of notation the following procedures are described for $H = 2$ agents and a single state variable $y_t \in \mathbb{R}^1$. However, the approach can analogously be extended to the general case of $H > 2$ agents and multiple states. We solve the social planner's problem using a collocation projection. For this we perform the usual transformation from an equilibrium described by the infinite sequences (with a time index t) to the equilibrium being described by functions of some state variable(s) x on a state space X. We denote the current exogenous state of the economy by y and the subsequent state in the next period by y' with the state space $Y \in \mathbb{R}^1$. $\underline{\lambda}_2$ denotes the current endogenous state of the Negishi weight and $\underline{\lambda}_2'$ denotes the corresponding state in the subsequent period with $\underline{\Lambda}_2 \in (0, 1)$.

We approximate the value functions of the two agents, $v^h(\underline{\lambda}_2, y), h = \{1, 2\}$, by two-dimensional cubic splines and we denote the approximated value functions by $\hat{v}^h(\underline{\lambda}_2, y)$. For the collocation projection we have to choose a set of collocation nodes $\{\underline{\lambda}_{2_k}\}_{k=0}^n$ and $\{y_l\}_{l=0}^m$ at which we evaluate $\hat{v}^h(\underline{\lambda}_2, y)$. The individual consumption shares only depend on the endogeneous state $\underline{\lambda}_{2_k}$. So in the following we show how to first solve for the individual consumption shares at the collocation nodes $s_k^h = s^h(\underline{\lambda}_{2_k})$ that are then used to solve for the value functions v^h and the dynamics of the endogenous state $\underline{\lambda}_2$.

Step 1: *Computing Optimal Consumption Allocations*

Equation (13) has to hold at each collocation node $\{\underline{\lambda}_{2_k}\}_{k=0}^n$:

$$\underline{\lambda}_{2_k}(1-\delta^2)\left(s_k^2\right)^{\rho^2-1} = (1-\underline{\lambda}_{2_k})(1-\delta^1)\left(s_k^1\right)^{\rho^1-1}.$$

Together with the market-clearing condition (11) we get

$$\underline{\lambda}_{2_k}(1-\delta^2)\left(s_k^2\right)^{\rho^2-1} = (1-\underline{\lambda}_{2_k})(1-\delta^1)\left(1-s_k^2\right)^{\rho^1-1}. \tag{51}$$

So for each node $\{\underline{\lambda}_{2_k}\}_{k=0}^n$ the optimal consumption choice s_k^2 can be computed by solving Equation (51) and s_k^1 is obtained by the market-clearing condition (11).[9] For the special case of $\rho^2 = \rho^1$ we can solve for s^2 as a function of $\underline{\lambda}_2$ analytically, and hence we don't have to solve the system of equations for each node.

Step 2: *Solving for the Value Function and the Dynamics of the Negishi Weights*

Solving for the value function is not as straight-forward, as it depends on the dynamics of the endogenous state $\underline{\lambda}_2$, which are unknown and follow Equation (14). We compute the expectation over the exogenous state by a Gauss-Quadrature with Q quadrature nodes. This implies that the values for y' at which we evaluate v^h are given by the quadrature rule. We denote the corresponding quadrature nodes by $\{y_{l,g}^{'}\}_{l=0,g=1}^{m,Q}$ and the weights by $\{\omega_g\}_{g=1}^{Q}$.[10] We can then solve Equation (14) for a given pair of collocation nodes $\{\underline{\lambda}_{2_k}, y_l\}_{k=0,l=0}^{n,m}$ and the corresponding quadrature nodes $\{y_{l,g}^{'}\}_{l=0,g=1}^{m,Q}$ to compute a vector $\vec{\underline{\lambda}}_2^{'}$ of size $(n+1)\times(m+1)\times Q$ that consists of the corresponding values $\underline{\lambda}_{2_{k,l,g}}^{'}$ for each node. For each $\underline{\lambda}_{2_{k,l,g}}^{'}$, Equation (14)

[9] Note that in the case of H agents we have to solve a system of $H-1$ equations that pin down the $H-1$ individual consumption choices $s^h \in \mathbb{H}^-$.

[10] Note that the quadrature nodes $\{\{y_{l,g}^{'}\}_{g=0}^{G}\}_{l=0}^{m}$ depend on the state today, $\{y_l\}_{l=0}^{m}$.

then reads

$$
\begin{aligned}
\underline{\lambda}'_{2_{k,l,g}} &= \frac{\underline{\lambda}_{2_k}\underline{\Pi}^2}{(1-\underline{\lambda}_{2_k})\underline{\Pi}^1 + \underline{\lambda}_{2_k}\underline{\Pi}^2} \\
\underline{\Pi}^h &= \delta^h e^{\rho^h \Delta c(y'_{l,g})} \left(\frac{v^h(\underline{\lambda}'_{2_{k,l,g}}, y'_{l,g}) e^{\Delta c(y'_{l,g})}}{R^h\left[v^h(\underline{\lambda}'_2, y') e^{\Delta c(y')} | \underline{\lambda}_{2_k}, y_l\right]} \right)^{\alpha^h - \rho^h} \frac{\mathrm{dP}^h(y'_{l,g}|y_l)}{\mathrm{dP}(y'_{l,g}|y_l)},
\end{aligned}
\tag{52}
$$

where

$$
R^h\left[v^h(\underline{\lambda}'_2, y') e^{\Delta c(y')} | \underline{\lambda}_{2_k}, y_l\right] = G_h^{-1}\left(E\left[G_h\left(v^h(\underline{\lambda}'_2, y') e^{\Delta c(y')}\right) \frac{\mathrm{dP}^h(y')}{\mathrm{dP}(y')} \middle| \underline{\lambda}_{2_k}, y_l \right] \right).
$$

Note that $\underline{\lambda}'_{2_{k,l,g}}$ depends on the full distribution of $\underline{\lambda}'_2$ through the expectation operator. By applying the Gauss-Quadrature to compute the expectation we get

$$
E\left[G_h\left(v^h(\underline{\lambda}'_2, y') e^{\Delta c(y')}\right) \frac{\mathrm{dP}^h(y')}{\mathrm{dP}(y')} \middle| \underline{\lambda}_{2_k}, y_l \right] \approx \sum_{g=1}^{Q} G_h\left(v^h(\underline{\lambda}'_{2_{k,l,g}}, y_{l,g}) e^{\Delta c(y_{l,g})}\right) \cdot \omega_g.
$$

So by computing the expectation with the quadrature rule, we do not need the full distribution of $\underline{\lambda}'_2$; instead we only have to evaluate v^h at those values $\underline{\lambda}'_{2_{k,l,g}}$ that can be obtained by solving (52) for each pair of collocation nodes $\{\underline{\lambda}_{2_k}, y_l\}_{k=0,l=0}^{n,m}$ and the corresponding quadrature nodes $\{y'_{l,g}\}_{l=0,g=1}^{m,G}$. So at the end we have a square system of equations with $(n+1)\times(m+1)\times G$ unknowns, $\underline{\lambda}'_{2_{k,l,g}}$, and as many equations (52) for each $\{k,l,g\}$.

The value function is in general not known so we have to compute it simultaneously when solving for $\underline{\lambda}'_{2_{k,l,g}}$. Plugging the approximation $\hat{v}^h(\underline{\lambda}_2, y)$ into the value function (12) yields

$$
\hat{v}^h(\underline{\lambda}_{2_k}, y_l) = \left[(1-\delta^h)\left(s_k^h\right)^{\rho^h} + \delta^h R^h\left(\hat{v}^h(\underline{\lambda}'_2, y') e^{\Delta c(y')} \middle| \underline{\lambda}_{2_k}, y_l\right) \right]^{\frac{1}{\rho^h}}. \tag{53}
$$

The collocation projection conditions require that the equation has to hold at each collocation node $\{\underline{\lambda}_{2_k}, y_l\}_{k=0,l=0}^{n,m}$. So we obtain a square system of equations with $(n+1)\times(m+1)\times 2$ equations (53) and as many unknowns for the spline interpolation at each collocation node, which we solve simultaneously with the system for $\underline{\lambda}'_{2_{k,l,g}}$ described above.

B.2 Properties of the Value Function

In the case of heterogeneous agents the approximation of the value function is a delicate computational task as an agent can vanish over time. The marginal utility of the agent for this limiting case is infinity, which makes it difficult to obtain accurate approximations for the value function close to the singularity. To obtain information about the properties of the singularity, we formally derive the limiting behavior of the value function for the special case of an economy with no uncertainty. We then include this information in the value function approximation for the stochastic economy. From Equation (13) we know that

$$s^2(\underline{\lambda}_2) = \left(\frac{1-\delta^1}{1-\delta^2}\right)^{-\psi^2} (\underline{\lambda}_2)^{\psi^2} (1-\underline{\lambda}_2)^{-\psi^2} (s^1(\underline{\lambda}_2))^{\frac{\psi^2}{\psi^1}}. \tag{54}$$

We are interested in the properties of $s^2(\underline{\lambda}_2)$ for $\underline{\lambda}_2$ close to 0. For $\underline{\lambda}_2 \approx 0$, agent 1 obtains all consumption so $s^1(\underline{\lambda}_2) \approx 1$ and the Negishi weight of the first agent becomes 1. Therefore, we obtain

$$s^2(\underline{\lambda}_2) \approx \left(\frac{1-\delta^1}{1-\delta^2}\right)^{\psi^2} (\underline{\lambda}_2)^{\psi^2} \tag{55}$$

for $\underline{\lambda}_2$ close to 0. The value function (39) for the deterministic economy at the steady state $y = y', \underline{\lambda}_2 = \underline{\lambda}_2'$ is given by

$$v^2(\underline{\lambda}_2, y) = s^2(\underline{\lambda}_2). \tag{56}$$

Inserting the behavior of $s^2(\underline{\lambda}_2)$ for $\underline{\lambda}_2$ close to 0, we obtain[11]

$$v^2(\underline{\lambda}_2) \approx \left(\frac{1-\delta^1}{1-\delta^2}\right)^{\psi^2} (\underline{\lambda}_2)^{\psi^2} \equiv \Upsilon^0(\underline{\lambda}_2). \tag{57}$$

We denote by $\Upsilon^0(\underline{\lambda}_2)$ the zero basis functions, which we add to the cubic spline value function approximation to obtain accurate approximations close to the singularity. We find that for all solutions reported in this paper, including the zero basis functions improves the accuracy of the solution. This concludes the description of the methodology for solving the heterogeneous agent model with recursive preferences.

B.3 Computational Details

For the projection method outlined above we need to choose certain collocation nodes. In this paper we use 17 uniform nodes for the $\underline{\lambda}^2$ dimension and 13 uniform nodes for the x_t dimension for the results with $\rho^2 = 0.975$ and $\rho^1 = 0.985$. For the results with $\rho^2 = 0.95$ and $\rho^1 = 0.985$, we use 51 uniform nodes for the $\underline{\lambda}^2$ dimension and 23 uniform nodes for the x_t dimension. For $\underline{\lambda}^2$ the minimum and maximum values are given by 0 and 1. For x_t we choose the approximation interval to cover ± 4 standard deviations around the unconditional mean

[11] For the first agent we obtain a similar expression for $\underline{\lambda}_2$ close to 1 given by $v^1(\underline{\lambda}_2) \approx \left(\frac{1-\delta^2}{1-\delta^1}\right)^{\psi^1} (1-\underline{\lambda}_2)^{\psi^1}$.

of the process. We approximate the value functions using two-dimensional cubic splines with not-a-knot end conditions. We provide the solver with additional information that we can formally derive for the limiting cases. For example, we know that for $\underline{\lambda}_t^2 = 1$ ($\underline{\lambda}_t^2 = 0$) agent 2 (1) consumes everything, so it corresponds to the representative-agent economy populated only by agent 2 (1). Hence, we require that the value function for these cases equals the value function for the corresponding representative-agent economy. We also know that for $\underline{\lambda}_t^2 = 0$ ($\underline{\lambda}_t^2 = 1$) the consumption of agent 2 (1) is 0 and hence the value function is also 0. As the shocks in the model are normally distributed, we compute the expectations over the exogenous states by Gauss–Hermite quadrature using 5 nodes for the shock in x_{t+1} and 3 nodes for the shock in Δc_{t+1}. Euler errors for the value function approximations evaluated on a 200×200 uniform grid for both states are less than 1×10^{-6}, suggesting a high accuracy of our results. We double-checked the accuracy by increasing the approximation interval as well as the number of collocation nodes, with no significant change in the results.

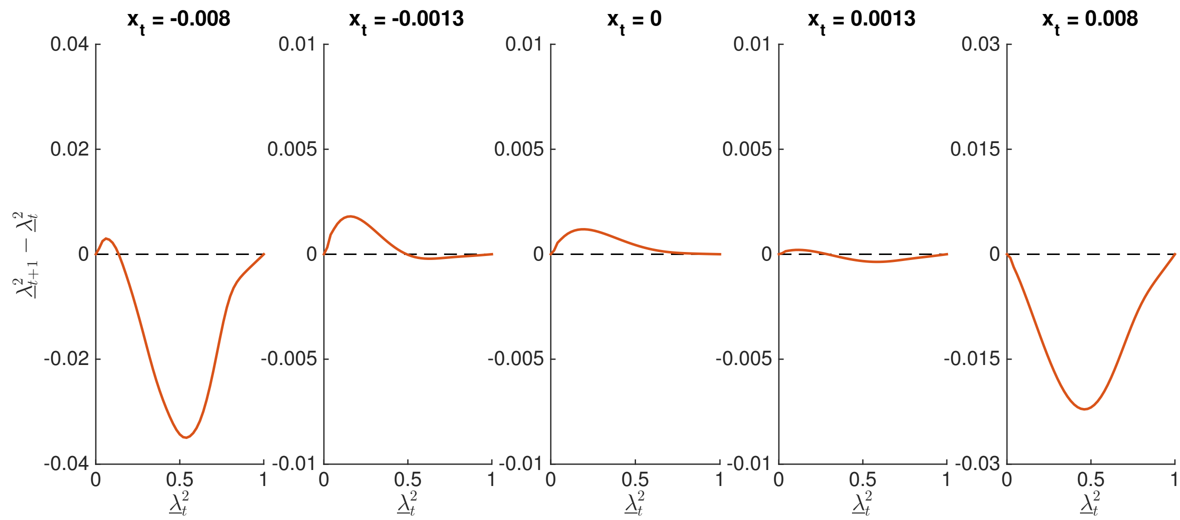
C Additional Results

Table 5: Equity Premia for Different Consumption Shares ($\rho_x = \rho_x^2$)

	$\boldsymbol{\rho_x^2 = 0.975}$		$\boldsymbol{\rho_x^2 = 0.95}$	
	s_t^2	Equity Premium	s_t^2	Equity Premium
Rep. Agent 1	0	6.49	0	6.50
0 Years	0.01	6.38	0.01	5.36
100 Years	0.3404	4.39	0.8147	0.49
200 Years	0.7249	3.33	0.8810	0.41
500 Years	0.9732	2.83	0.9388	0.34
Rep. Agent 2	1	2.75	1	0.25

The table shows the annualized equity premium for a specific consumption share $s_t^2 = \bar{s}$. The premium is reported for the equilibrium allocations after 0, 100, 200, and 500 years of simulated data assuming an initial share of $s_0^2 = 0.01$ (see Table 2). Agent 1 believes that $\rho_x^1 = 0.985$ and agent 2 has the correct belief ($\rho_x = \rho_x^2$). The left panel depicts the case for $\rho_x^2 = 0.975$ and the right panel for $\rho_x^2 = 0.95$.

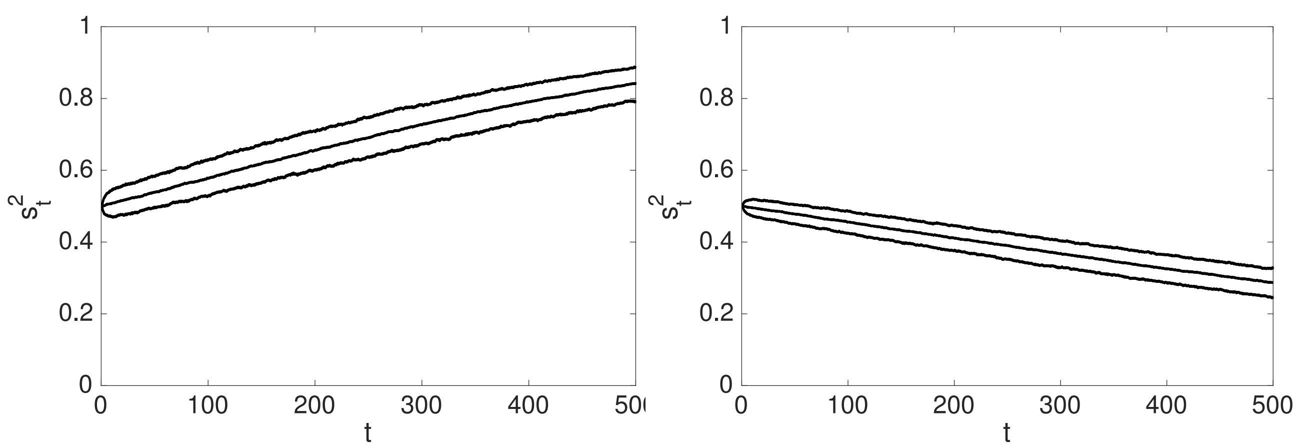
Figure 12: Changes in the Wealth Distribution—The Epstein–Zin Case 2 ($\rho_x^2 = 0.95$)



The figure shows the change in the optimal weights $\underline{\lambda}_{t+1}^2 - \underline{\lambda}_t^2$ as a function of $\underline{\lambda}_t^2$. From left to right, the change is shown for $x_t = \{-0.008, -0.0013, 0, 0.0013, 0.008\}$ ($\pm$ 4 standard deviations). The red line shows the average over all shocks in x_{t+1}. Calibration with $\rho_x = \rho_x^1 = 0.985$ and $\rho_x^2 = 0.95$.

Figure 13: Consumption Shares for Heterogeneous Preferences—Simulation 2

(a) $\gamma^2 = 10, \psi^2 = 0.8$ (b) $\gamma^2 = 10, \psi^2 = 2.2$



The figure shows the median, 5%, and 95% quantile paths of the consumption share of agent 2 for 1'000 samples each consisting of 500 years of simulated data. The investors have identical beliefs about ρ_x with $\rho_x = \rho_x^1 = \rho_x^2 = 0.985$, but differ with respect to their preference parameters. Agent 1 has a risk aversion of $\gamma^1 = 10$ and an EIS of $\psi^1 = 1.5$ and the different panels show the results for different preference parameters of the second investor.