Fuzzy Matrices and Fuzzy Markov Chains

Sy-Ming Guu

Chang Gung University

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Outline

Fuzzy Matrices

- Fuzzy matrices: $\mathbb{F} = \{[a_{ij}] : a_{ij} \in [0,1]\}$
  Let $A, B \in \mathbb{F}$. Define

$$C = A \otimes B \text{ by } C_{ij} = \max_t \{a_{it} \otimes b_{tj}\},$$

where $\otimes$: fuzzy algebraic operation.

Fuzzy matrices (conti.)

Example ($\otimes \equiv$ "min".)

$$C_{ij} = \max_t \{\min(a_{it}, b_{tj})\}$$

Fuzzy matrices (conti.)

Example ($\otimes \equiv$ "min".)

$$C_{ij} = \max_t \{\min(a_{it}, b_{tj})\}$$

Example ($\otimes \equiv$ "product" (as regular).)

$$C_{ij} = \max_t \{a_{it} \cdot b_{tj}\}$$

Fuzzy matrices (conti.)

Example ($\otimes \equiv$ "t-norm".)

$$C_{ij} = \max_t \{T(a_{it}, b_{tj})\}.$$

Recall: t-norm, $T : [0,1] \times [0,1] \to [0,1]$ satisfying

(i) $T(\alpha, \beta) = T(\beta, \alpha)$ commutative

(ii) $T(\alpha, T(\beta, \gamma)) = T(T(\alpha, \beta), \gamma)$ associative

(iii) $T(\alpha, \beta) \le T(\alpha, \gamma)$ whenever $\beta \le \gamma$

(iv) $T(\alpha, 1) = \alpha$

Fuzzy matrices (conti.)

Example ($\otimes =$ "average operation".)

$$C_{ij} = \max_t \{ \frac{a_{it} + b_{tj}}{2} \}.$$

Fuzzy matrices (conti.)

Example ($\otimes =$ "average operation".)

$$C_{ij} = \max_t\{\frac{a_{it} + b_{tj}}{2}\}.$$

Example ($\otimes =$ "general average operation".)

$$C_{ij} = \max_t\{\lambda a_{it} + (1-\lambda)b_{tj}\}, \text{ where } 0 < \lambda < 1.$$

or

$$C_{ij} = \max_t\{(\lambda a_{it}^p + (1-\lambda)b_{tj}^p)^{\frac{1}{p}}\}, \text{ where } -\infty < p < \infty.$$

Fuzzy matrices (conti.)

Example ($\otimes =$ "convex combination of two operations".)

For instance:

(i) $C_{ij} = \max_t\{\lambda \min(a_{it}, b_{tj}) + (1-\lambda)\frac{a_{it}+b_{tj}}{2}\}$,
where $0 < \lambda < 1$.

(ii) $C_{ij} = \max_t\{\lambda \min(a_{it}, b_{tj}) + (1-\lambda)(\phi a_{it}^p + (1-\phi)b_{tj}^p)^{\frac{1}{p}}\}$,
where $-\infty < p < \infty, 0 < \lambda < 1$.

(iii) $C_{ij} = \max_t\{\lambda T(a_{it}, b_{tj}) + (1-\lambda)\frac{a_{it}+b_{tj}}{2}\}$.

We may have more than dozens of instances.

Powers of a fuzzy matrix

- We consider $A^n_{\otimes} = A \otimes (A \otimes \cdots \otimes (A \otimes A))$.
- The "limiting" behaviour of $A^n_{\otimes}$ depends heavily on the $\otimes$.
  - We are interested in the "convergence" situation.

Powers of a fuzzy matrix (conti.)

- A very original result: Thomason 1976 in JMAA. The sequence $\{A_{\otimes}^{n}\}$, where $\otimes = \min$, displays two kinds of behaviours
  - i Convergence in a **finite** power: $\exists k \in \mathbf{N} \ni$

$$A_{\otimes}^{k} = A_{\otimes}^{k+1} = \cdots$$

  - ii Oscillation: There exist c_0, p such that

$$A_{\otimes}^{c} = A_{\otimes}^{c+kp}, k \in \mathbf{N}, c \geq c_0 \geq 1$$

    If $p = 1$, we have a convergence. The minimal such p is called period.

The max-product powers of a fuzzy matrix

- Major reference: C.-T Pang and Sy-Ming Guu, LAA (2001).

The max-product powers of a fuzzy matrix (conti.)

- Numerical example to contrast with the max-min powers. Let

$$\mathbf{R} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We have, under <u>max-product</u> operation,

$$\mathbf{R}_{\otimes}^{2k+1} = \begin{bmatrix} 0.1^{2k+1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{R}_{\otimes}^{2k} = \begin{bmatrix} 0.1^{2k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the powers of such matrices may "Oscillate" with an <u>infinite</u> period.

Powers of a max-product fuzzy matrix

Key features and a powerful characterization.

- may need infinite steps to "settle down".
- "Motivation": asymptotic period.
  A sequence $\{A_l : l \in \mathbb{N}\}$ of matrices in $\mathbb{R}^{n\times n}$ is asymptotically periodic if $\lim_{k\to\infty} A_{j+kp} = \overline{A}_j$ exists for all $j = 1, 2, \ldots, p$.
  The minimal such p is called asymptotic period p. If $p = 1$, then we have a convergent sequence.

How to detect the asymptotic period?

- First decompose a fuzzy matrix $\mathbf{R} = \overline{\mathbf{R}} \oplus \underline{\mathbf{R}}$, where $[\overline{\mathbf{R}}]_{st} = \begin{cases} 1 & \text{if } \mathbf{R}_{st} = 1 \\ 0 & \text{otherwise} \end{cases}$; $[\underline{\mathbf{R}}]_{st} = \begin{cases} 0 & \text{if } \mathbf{R}_{st} = 1 \\ \mathbf{R}_{st} & \text{otherwise} \end{cases}$
- Note that $\overline{\mathbf{R}}$ is a Boolean matrix. It is very easy to calculate its period.
- [Pang and Guu]: $\overline{\mathbf{R}}$ has period p if and only if the sequence of powers of $\mathbf{R}_{\otimes}$ has asymptotic period P.

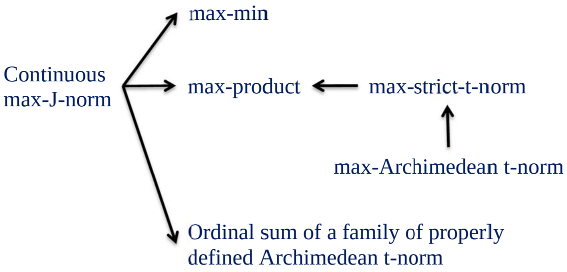
Numerical example for the asymptotic period

Recall:

$$\mathbf{R} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \triangleq \overline{\mathbf{R}} \oplus \underline{\mathbf{R}}$$

Obviously, the Boolean matrix $\overline{\mathbf{R}}$ has period 2. And we notice from $\mathbf{R}_{\otimes}^{2k+1}$ and $\mathbf{R}_{\otimes}^{2k}$ that the sequence $\{\mathbf{R}^{l} : l \in \mathbb{N}\}$ has the asymptotic period 2.

A more complete picture



Note that: Continuous Archimedean t-norm J if T is a continuous t-norm and $0 < T(\alpha, \alpha) < \alpha' \ \forall \alpha \in (0, 1)$

Fuzzy Markov chain (Avrachenkov and Sanchez 2002)

Definition

A (finite) fuzzy set on $S = \{1, 2, \ldots, n\}$ is defined by a mapping $x : S \to [0, 1]$ represented by a vector $x = (x_1, \ldots, x_n)$, where $x_i = x(i)$. The set of all fuzzy sets on S is denoted by $\mathscr{F}(S)$.

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Definition

A fuzzy relation ρ is defined as a fuzzy set on $S \times S$. P is represented by a matrix (P_{ij}), where $P_{ij} = P[i, j]$.

Fuzzy Markov chain (Avrachenkov and Sanchez) (conti.)

Definition

At each instant $t, t = 0, 1, 2, \ldots$ the state of the system is described by the fuzzy set $x^{(t)} \in \mathscr{F}(S)$. The transition law of the fuzzy markov chain is given by P as follows

$$x_j^{(t+1)} = \max_{i \in S}\{\min\{x_i^{(t)}, P_{ij}\}\} \quad \forall j \in S$$

Recall: $x_j^{(t+1)} = \sum_{i=1}^{n} x_i^{(t)} P_{ij}$

Ergodic Property: aperiodic and identical rows in the limiting matrix

- Motivation: Ergodic Property $\Longrightarrow$ ranking
- Not every classical Markov chain has this property. (Sufficient condition: aperiodic, recurrent)
- We have reviewed the powers of a fuzzy matrix of course, <u>convergence does Not imply ergodicity.</u>
- Open question: find conditions to ensure the ergodicity for a max-min fuzzy Markov chain. (Avrachenkov and Sanchez 2002)

Ranking by the ergodicity (1)

- classic Markov chain $\Longrightarrow$ Partial capability (by some sufficient conditions)
- fuzzy Markov chain $\Longrightarrow$ depends on the $\otimes$ operation.

  $$\left.\begin{array}{l}\text{(i) max-min type} \Rightarrow \text{unknown}\\ \text{(ii) max-product type} \Rightarrow \text{unknown}\\ \text{(iii) max-T-norm type} \Rightarrow \text{unknown}\end{array}\right\}$$

  We only known some conditions for the convergence.
- We need to find other operations. And we are very Lucky...

Ranking and max-average operation

Recap: $C_{ij} = \max_t \{\frac{a_{it}+b_{tj}}{2}\}$, where $C = A \otimes B$

- Properties of the powers of a max-average fuzzy matrix. (Lur, Wu and Guu 2007, FSS)
  1. The sequence of powers of any max-average fuzzy matrix is always convergent.
  2. The limiting matrix has identical rows.

Two extreme cases

Example

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Both classical Markov chain and max-min fuzzy Markov chain fail to provide reasonable ranking for ① and ②.

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Both classical Markov chain and max-min fuzzy Markov chain fail to provide reasonable ranking for ① and ②.

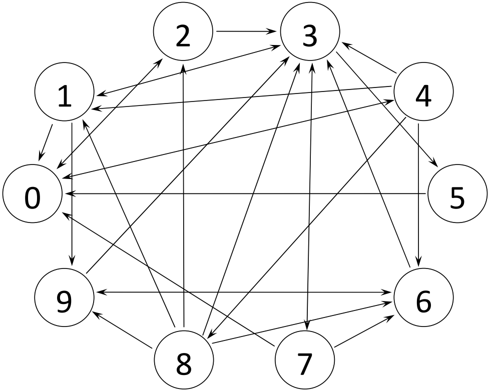
Example

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad ① \overset{1,0}{\rightleftarrows} ②$$

Web page ranking

- Google: ergodicity from classical Markov chain area.
- Consider the following web structure.
- Note: $M \leftarrow dM + (1-d)I \ (d = 0.85)$

Sample problem's Graph



A sample graph with 10 nodes and 28 links

Original Case 1: $d = 1$

$$M = \begin{bmatrix} 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 1/5 & 1/5 & 0 & 1/5 & 0 & 0 & 1/5 & 0 & 1/5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/5 & 1/5 & 1/5 & 0 & 0 & 1/5 & 0 & 0 & 1/5 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \end{bmatrix}$$

Original Case 1: $d = 1$ (conti.)

$$M^2 = \begin{bmatrix}
1/2 & 7/20 & 1/4 & 1/2 & 1/4 & 1/4 & 7/20 & 1/4 & 7/20 & 1/4 \\
1/2 & 1/3 & 5/12 & 5/12 & 5/12 & 1/3 & 5/12 & 1/3 & 1/6 & 1/4 \\
1/2 & 5/12 & 1/2 & 1/4 & 1/2 & 5/12 & 1/4 & 5/12 & 1/4 & 1/4 \\
2/3 & 1/6 & 1/4 & 1/3 & 1/4 & 1/6 & 1/3 & 1/6 & 1/6 & 1/3 \\
1/2 & 4/15 & 7/20 & 7/20 & 7/20 & 4/15 & 1/4 & 4/15 & 1/10 & 7/20 \\
1/2 & 1/2 & 3/4 & 1/2 & 3/4 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 5/12 & 1/4 & 1/2 & 1/4 & 5/12 & 1/2 & 5/12 & 1/4 & 1/4 \\
1/2 & 1/3 & 5/12 & 5/12 & 5/12 & 1/3 & 1/4 & 1/3 & 1/6 & 5/12 \\
1/2 & 4/15 & 1/4 & 7/20 & 1/4 & 4/15 & 7/20 & 4/15 & 1/10 & 7/20 \\
1/2 & 5/12 & 1/4 & 1/2 & 1/4 & 5/12 & 1/4 & 5/12 & 1/4 & 1/2
\end{bmatrix}$$

Original Case 1: $d = 1$ (conti.)

$$M^3 = \begin{bmatrix} 5/8 & 5/12 & 1/2 & 17/40 & 1/2 & 5/12 & 3/8 & 5/12 & 1/4 & 17/40 \\ 2/3 & 3/8 & 1/2 & 11/24 & 1/2 & 3/8 & 3/8 & 3/8 & 4/13 & 11/24 \\ 17/24 & 7/20 & 1/2 & 1/2 & 1/2 & 7/24 & 3/8 & 7/24 & 7/20 & 3/8 \\ 7/12 & 1/3 & 7/12 & 5/12 & 7/12 & 1/3 & 5/12 & 1/3 & 1/3 & 5/12 \\ 19/30 & 14/41 & 1/2 & 17/40 & 1/2 & 14/41 & 17/40 & 14/41 & 11/40 & 3/8 \\ 3/4 & 19/40 & 1/2 & 5/8 & 1/2 & 5/12 & 1/2 & 5/12 & 19/40 & 1/2 \\ 17/24 & 5/12 & 1/2 & 1/2 & 1/2 & 5/12 & 3/8 & 5/12 & 1/4 & 1/2 \\ 2/3 & 3/8 & 1/2 & 11/24 & 1/2 & 3/8 & 11/24 & 3/8 & 4/13 & 3/8 \\ 19/30 & 14/41 & 1/2 & 17/40 & 1/2 & 14/41 & 17/40 & 14/41 & 1/4 & 17/40 \\ 17/24 & 5/12 & 1/2 & 1/2 & 1/2 & 5/12 & 1/2 & 5/12 & 1/4 & 3/8 \end{bmatrix}$$

Original Case 1: $d = 1$ (conti.)

$$M^{15} = \begin{bmatrix} 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \\ 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \\ 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \\ 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \\ 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \\ 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \\ 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \\ 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \\ 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \\ 13/18 & 4/9 & 11/18 & 5/9 & 11/18 & 4/9 & 1/2 & 4/9 & 15/37 & 1/2 \end{bmatrix}$$

Original Case 2: $d = 0.85$

$$M^1 = \begin{bmatrix}
0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\
0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.2983 \\
0.4400 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\
0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.2983 & 0.0150 & 0.0150 \\
0.1850 & 0.1850 & 0.0150 & 0.1850 & 0.0150 & 0.0150 & 0.1850 & 0.0150 & 0.1850 & 0.0150 \\
0.8650 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\
0.0150 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.4400 \\
0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 \\
0.0150 & 0.1850 & 0.1850 & 0.1850 & 0.0150 & 0.0150 & 0.1850 & 0.0150 & 0.0150 & 0.1850 \\
0.0150 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150
\end{bmatrix}$$

Original Case 2: $d = 0.85$ (conti.)

$$M^2 = \begin{bmatrix} 0.4400 & 0.3125 & 0.2275 & 0.4400 & 0.2275 & 0.2275 & 0.3125 & 0.2275 & 0.3125 & 0.2275 \\ 0.4400 & 0.2983 & 0.3692 & 0.3692 & 0.3692 & 0.2983 & 0.3692 & 0.2983 & 0.1567 & 0.2275 \\ 0.4400 & 0.3692 & 0.4400 & 0.2275 & 0.4400 & 0.3692 & 0.2275 & 0.3692 & 0.2275 & 0.2275 \\ 0.5817 & 0.1567 & 0.2275 & 0.2983 & 0.2275 & 0.1567 & 0.2983 & 0.1567 & 0.1567 & 0.2983 \\ 0.4400 & 0.2417 & 0.3125 & 0.3125 & 0.3125 & 0.2417 & 0.2275 & 0.2417 & 0.1000 & 0.3125 \\ 0.4400 & 0.4400 & 0.6525 & 0.4400 & 0.6525 & 0.4400 & 0.4400 & 0.4400 & 0.4400 & 0.4400 \\ 0.4400 & 0.3692 & 0.2275 & 0.4400 & 0.2275 & 0.3692 & 0.4400 & 0.3692 & 0.2275 & 0.2275 \\ 0.4400 & 0.2983 & 0.3692 & 0.3692 & 0.3692 & 0.2983 & 0.2275 & 0.2983 & 0.1567 & 0.3692 \\ 0.4400 & 0.2417 & 0.2275 & 0.3125 & 0.2275 & 0.2417 & 0.3125 & 0.2417 & 0.1000 & 0.3125 \\ 0.4400 & 0.3692 & 0.2275 & 0.4400 & 0.2275 & 0.3692 & 0.2275 & 0.3692 & 0.2275 & 0.4400 \end{bmatrix}$$

Original Case 2: $d = 0.85$ (conti.)

$$M^3 = \begin{bmatrix} 0.5462 & 0.3692 & 0.4400 & 0.3762 & 0.4400 & 0.3692 & 0.3338 & 0.3692 & 0.2275 & 0.3762 \\ 0.5817 & 0.3337 & 0.4400 & 0.4046 & 0.4400 & 0.3337 & 0.3338 & 0.3337 & 0.2771 & 0.4046 \\ 0.6171 & 0.3125 & 0.4400 & 0.4400 & 0.4400 & 0.2629 & 0.3338 & 0.2629 & 0.3125 & 0.3338 \\ 0.5108 & 0.2983 & 0.5108 & 0.3692 & 0.5108 & 0.2983 & 0.3692 & 0.2983 & 0.2983 & 0.3692 \\ 0.5533 & 0.3054 & 0.4400 & 0.3762 & 0.4400 & 0.3054 & 0.3762 & 0.3054 & 0.2488 & 0.3338 \\ 0.6525 & 0.4188 & 0.4400 & 0.5462 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.4188 & 0.4400 \\ 0.6171 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.3338 & 0.3692 & 0.2275 & 0.4400 \\ 0.5817 & 0.3337 & 0.4400 & 0.4046 & 0.4400 & 0.3337 & 0.4046 & 0.3337 & 0.2771 & 0.3338 \\ 0.5533 & 0.3054 & 0.4400 & 0.3762 & 0.4400 & 0.3054 & 0.3762 & 0.3054 & 0.2275 & 0.3762 \\ 0.6171 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.2275 & 0.3338 \end{bmatrix}$$

Original Case 2: $d = 0.85$ (conti.)

$$M^{20} = \begin{bmatrix} 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3597 & 0.4400 \end{bmatrix}$$

Case 1:

Please let A(5,1)=1/4; A(5,2)=1/4; A(5, 7)=1/4; A(5, 9)=1/4. The rest elements in this row are zero. Please then use this new matrix A to compute the powers of M.

Example

$$M^1 = \begin{bmatrix} 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\ 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.2983 \\ 0.4400 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\ 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.2983 & 0.0150 & 0.0150 \\ 0.2275 & 0.2275 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.2275 & 0.0150 & 0.2275 & 0.0150 \\ 0.8650 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\ 0.0150 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.4400 \\ 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 \\ 0.0150 & 0.1850 & 0.1850 & 0.1850 & 0.0150 & 0.0150 & 0.1850 & 0.0150 & 0.0150 & 0.1850 \\ 0.0150 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 \end{bmatrix}$$

Case 1 (conti.):

Example

$$M^2 = \begin{bmatrix} 0.4400 & 0.3338 & 0.2275 & 0.4400 & 0.2275 & 0.2275 & 0.3338 & 0.2275 & 0.3338 & 0.2275 \\ 0.4400 & 0.2983 & 0.3692 & 0.3692 & 0.3692 & 0.2983 & 0.3692 & 0.2983 & 0.1567 & 0.2275 \\ 0.4400 & 0.3692 & 0.4400 & 0.2275 & 0.4400 & 0.3692 & 0.2275 & 0.3692 & 0.2275 & 0.2275 \\ 0.5817 & 0.1567 & 0.2275 & 0.2983 & 0.2275 & 0.1567 & 0.2983 & 0.1567 & 0.1567 & 0.2983 \\ 0.4400 & 0.2063 & 0.3338 & 0.3338 & 0.3338 & 0.1567 & 0.2275 & 0.1567 & 0.1213 & 0.3338 \\ 0.4400 & 0.4400 & 0.6525 & 0.4400 & 0.6525 & 0.4400 & 0.4400 & 0.4400 & 0.4400 & 0.4400 \\ 0.4400 & 0.3692 & 0.2275 & 0.4400 & 0.2275 & 0.3692 & 0.4400 & 0.3692 & 0.2275 & 0.2275 \\ 0.4400 & 0.2983 & 0.3692 & 0.3692 & 0.3692 & 0.2983 & 0.2275 & 0.2983 & 0.1567 & 0.3692 \\ 0.4400 & 0.2417 & 0.2275 & 0.3125 & 0.2275 & 0.2417 & 0.3125 & 0.2417 & 0.1213 & 0.3125 \\ 0.4400 & 0.3692 & 0.2275 & 0.4400 & 0.2275 & 0.3692 & 0.2275 & 0.3692 & 0.2275 & 0.4400 \end{bmatrix}$$

Case 1 (conti.):

Example

$$M^3 = \begin{bmatrix} 0.5462 & 0.3692 & 0.4400 & 0.3869 & 0.4400 & 0.3692 & 0.3338 & 0.3692 & 0.2275 & 0.3869 \\ 0.5817 & 0.3337 & 0.4400 & 0.4046 & 0.4400 & 0.3337 & 0.3338 & 0.3337 & 0.2983 & 0.4046 \\ 0.6171 & 0.3338 & 0.4400 & 0.4400 & 0.4400 & 0.2629 & 0.3338 & 0.2629 & 0.3338 & 0.3338 \\ 0.5108 & 0.2983 & 0.5108 & 0.3692 & 0.5108 & 0.2983 & 0.3692 & 0.2983 & 0.2983 & 0.3692 \\ 0.5108 & 0.3160 & 0.4400 & 0.3869 & 0.4400 & 0.3160 & 0.3869 & 0.3160 & 0.2806 & 0.3338 \\ 0.6525 & 0.4400 & 0.4400 & 0.5462 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.4400 & 0.4400 \\ 0.6171 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.3338 & 0.3692 & 0.2275 & 0.4400 \\ 0.5817 & 0.3337 & 0.4400 & 0.4046 & 0.4400 & 0.3337 & 0.4046 & 0.3337 & 0.2983 & 0.3338 \\ 0.5533 & 0.3054 & 0.4400 & 0.3762 & 0.4400 & 0.3054 & 0.3762 & 0.3054 & 0.2275 & 0.3762 \\ 0.6171 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.2275 & 0.3338 \end{bmatrix}$$

Case 1 (conti.):

Example

$$M^{20} = \begin{bmatrix} 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \\ 0.6289 & 0.3928 & 0.5344 & 0.4872 & 0.5344 & 0.3928 & 0.4400 & 0.3928 & 0.3810 & 0.4400 \end{bmatrix}$$

Case 2:

Please let A(6,1)=0 and the rest elements in this row (row 6) are all zero as well. (Hence A has a zero row in its 6th row.) And compute the powers of M.

Example

$$M^1 = \begin{bmatrix} 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\ 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.2983 \\ 0.4400 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\ 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.2983 & 0.0150 & 0.0150 \\ 0.1850 & 0.1850 & 0.0150 & 0.1850 & 0.0150 & 0.0150 & 0.1850 & 0.0150 & 0.1850 & 0.0150 \\ 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\ 0.0150 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.4400 \\ 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 \\ 0.0150 & 0.1850 & 0.1850 & 0.1850 & 0.0150 & 0.0150 & 0.1850 & 0.0150 & 0.0150 & 0.1850 \\ 0.0150 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 \end{bmatrix}$$

Case 2 (conti.):

Example

$$M^2 = \begin{bmatrix} 0.4400 & 0.3125 & 0.2275 & 0.4400 & 0.2275 & 0.2275 & 0.3125 & 0.2275 & 0.3125 & 0.2275 \\ 0.2275 & 0.2983 & 0.3692 & 0.3692 & 0.3692 & 0.2983 & 0.3692 & 0.2983 & 0.1567 & 0.2275 \\ 0.2275 & 0.3692 & 0.4400 & 0.2275 & 0.4400 & 0.3692 & 0.2275 & 0.3692 & 0.2275 & 0.2275 \\ 0.2983 & 0.1567 & 0.2275 & 0.2983 & 0.2275 & 0.1567 & 0.2983 & 0.1567 & 0.1567 & 0.2983 \\ 0.2417 & 0.2417 & 0.3125 & 0.3125 & 0.3125 & 0.2417 & 0.2275 & 0.2417 & 0.1000 & 0.3125 \\ 0.2275 & 0.1567 & 0.2275 & 0.2275 & 0.2275 & 0.1567 & 0.2275 & 0.1567 & 0.1000 & 0.2275 \\ 0.2275 & 0.3692 & 0.2275 & 0.4400 & 0.2275 & 0.3692 & 0.4400 & 0.3692 & 0.2275 & 0.2275 \\ 0.2275 & 0.2983 & 0.3692 & 0.3692 & 0.3692 & 0.2983 & 0.2275 & 0.2983 & 0.1567 & 0.3692 \\ 0.3125 & 0.2417 & 0.2275 & 0.3125 & 0.2275 & 0.2417 & 0.3125 & 0.2417 & 0.1000 & 0.3125 \\ 0.2275 & 0.3692 & 0.2275 & 0.4400 & 0.2275 & 0.3692 & 0.2275 & 0.3692 & 0.2275 & 0.4400 \end{bmatrix}$$

Case 2 (conti.):

Example

$$M^3 = \begin{bmatrix} 0.3338 & 0.3692 & 0.4400 & 0.3762 & 0.4400 & 0.3692 & 0.3338 & 0.3692 & 0.2275 & 0.3762 \\ 0.4046 & 0.3337 & 0.3338 & 0.4046 & 0.3338 & 0.3337 & 0.3338 & 0.3337 & 0.2771 & 0.4046 \\ 0.4400 & 0.3125 & 0.3338 & 0.4400 & 0.3338 & 0.2629 & 0.3338 & 0.2629 & 0.3125 & 0.3338 \\ 0.3338 & 0.2983 & 0.3692 & 0.3692 & 0.3692 & 0.2983 & 0.3692 & 0.2983 & 0.2063 & 0.3692 \\ 0.3762 & 0.3054 & 0.3408 & 0.3762 & 0.3408 & 0.3054 & 0.3762 & 0.3054 & 0.2488 & 0.3338 \\ 0.3338 & 0.2629 & 0.3338 & 0.3338 & 0.3338 & 0.2629 & 0.3338 & 0.2629 & 0.2063 & 0.3338 \\ 0.3338 & 0.3692 & 0.3338 & 0.4400 & 0.3338 & 0.3692 & 0.3338 & 0.3692 & 0.2275 & 0.4400 \\ 0.4046 & 0.3337 & 0.3338 & 0.4046 & 0.3338 & 0.3337 & 0.4046 & 0.3337 & 0.2771 & 0.3338 \\ 0.3338 & 0.3054 & 0.3762 & 0.3762 & 0.3762 & 0.3054 & 0.3762 & 0.3054 & 0.2063 & 0.3762 \\ 0.3338 & 0.3692 & 0.3338 & 0.4400 & 0.3338 & 0.3692 & 0.4400 & 0.3692 & 0.2275 & 0.3338 \end{bmatrix}$$

Case 2 (conti.):

Example

$$M^{20} = \begin{bmatrix} 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \end{bmatrix}$$

Case 3:

Fuzzy Matrices and Fuzzy Markov Chains

Guu

Fuzzy Matrices

Powers of a fuzzy matrix

Please let A(6,1)=1/2; A(6,3)=1/2, and the rest of elements in this row are all zero. And compute the powers of M.

Example

$$M^1 = \begin{bmatrix} 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\ 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.2983 \\ 0.4400 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\ 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.2983 & 0.0150 & 0.0150 \\ 0.1850 & 0.1850 & 0.0150 & 0.1850 & 0.0150 & 0.0150 & 0.1850 & 0.0150 & 0.1850 & 0.0150 \\ 0.4400 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 \\ 0.0150 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.0150 & 0.4400 \\ 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.2983 & 0.0150 & 0.0150 & 0.0150 \\ 0.0150 & 0.1850 & 0.1850 & 0.1850 & 0.0150 & 0.0150 & 0.1850 & 0.0150 & 0.0150 & 0.1850 \\ 0.0150 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.4400 & 0.0150 & 0.0150 & 0.0150 \end{bmatrix}$$

Case 3 (conti.):

Example

$$M^2 = \begin{bmatrix} 0.4400 & 0.3125 & 0.2275 & 0.4400 & 0.2275 & 0.2275 & 0.3125 & 0.2275 & 0.3125 & 0.2275 \\ 0.2275 & 0.2983 & 0.3692 & 0.3692 & 0.3692 & 0.2983 & 0.3692 & 0.2983 & 0.1567 & 0.2275 \\ 0.2275 & 0.3692 & 0.4400 & 0.2275 & 0.4400 & 0.3692 & 0.2275 & 0.3692 & 0.2275 & 0.2275 \\ 0.3692 & 0.1567 & 0.3692 & 0.2983 & 0.2275 & 0.1567 & 0.2983 & 0.1567 & 0.1567 & 0.2983 \\ 0.2417 & 0.2417 & 0.3125 & 0.3125 & 0.3125 & 0.2417 & 0.2275 & 0.2417 & 0.1000 & 0.3125 \\ 0.4400 & 0.2275 & 0.4400 & 0.4400 & 0.4400 & 0.2275 & 0.2275 & 0.2275 & 0.2275 & 0.2275 \\ 0.2275 & 0.3692 & 0.2275 & 0.4400 & 0.2275 & 0.3692 & 0.4400 & 0.3692 & 0.2275 & 0.2275 \\ 0.2275 & 0.2983 & 0.3692 & 0.3692 & 0.3692 & 0.2983 & 0.2275 & 0.2983 & 0.1567 & 0.3692 \\ 0.3125 & 0.2417 & 0.2275 & 0.3125 & 0.2275 & 0.2417 & 0.3125 & 0.2417 & 0.1000 & 0.3125 \\ 0.2275 & 0.3692 & 0.2275 & 0.4400 & 0.2275 & 0.3692 & 0.2275 & 0.3692 & 0.2275 & 0.4400 \end{bmatrix}$$

Case 3 (conti.):

Example

$$M^3 = \begin{bmatrix} 0.3338 & 0.3692 & 0.4400 & 0.3762 & 0.4400 & 0.3692 & 0.3338 & 0.3692 & 0.2275 & 0.3762 \\ 0.4046 & 0.3337 & 0.3692 & 0.4046 & 0.3338 & 0.3337 & 0.3338 & 0.3337 & 0.2771 & 0.4046 \\ 0.4400 & 0.3125 & 0.4046 & 0.4400 & 0.3338 & 0.2629 & 0.3338 & 0.2629 & 0.3125 & 0.3338 \\ 0.4046 & 0.2983 & 0.4046 & 0.4046 & 0.4046 & 0.2983 & 0.3692 & 0.2983 & 0.2063 & 0.3692 \\ 0.3762 & 0.3054 & 0.3408 & 0.3762 & 0.3408 & 0.3054 & 0.3762 & 0.3054 & 0.2488 & 0.3338 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.3338 & 0.3692 & 0.3125 & 0.3338 \\ 0.4046 & 0.3692 & 0.4046 & 0.4400 & 0.3338 & 0.3692 & 0.3338 & 0.3692 & 0.2275 & 0.4400 \\ 0.4046 & 0.3337 & 0.3692 & 0.4046 & 0.3338 & 0.3337 & 0.4046 & 0.3337 & 0.2771 & 0.3338 \\ 0.3408 & 0.3054 & 0.3762 & 0.3762 & 0.3762 & 0.3054 & 0.3762 & 0.3054 & 0.2063 & 0.3762 \\ 0.4046 & 0.3692 & 0.4046 & 0.4400 & 0.3338 & 0.3692 & 0.4400 & 0.3692 & 0.2275 & 0.3338 \end{bmatrix}$$

Case 3 (conti.):

Example

$$M^{20} = \begin{bmatrix} 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \\ 0.4400 & 0.3692 & 0.4400 & 0.4400 & 0.4400 & 0.3692 & 0.4400 & 0.3692 & 0.3125 & 0.4400 \end{bmatrix}$$